FOURIER TRANSFORMS OF SEMISIMPLE ORBITAL INTEGRALS ON THE LIE ALGEBRA OF SL_2

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Abstract. The Harish-Chandra–Howe local character expansion expresses the characters of reductive, p-adic groups in terms of Fourier transforms of nilpotent orbital integrals on their Lie algebras, and Murnaghan–Kirillov theory expresses many characters of reductive, p-adic groups in terms of Fourier transforms of semisimple orbital integrals (also on their Lie algebras). In many cases, the evaluation of these Fourier transforms seems intractable; but, for SL_2, the nilpotent orbital integrals have already been computed [17, Appendix A]. In this paper, we use a variant of Huntsinger's integral formula, and the theory of p-adic special functions, to compute semisimple orbital integrals.

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1. Introduction

1.1. **History.** Harish-Chandra's p-adic Lefschetz principle suggests that results in real harmonic analysis should have analogues in p-adic harmonic analysis. This principle has had too many successes to list, but it is interesting that the paths to results in the Archimedean and non-Archimedean settings are often different. One striking manifestation of this is that the characters for the discrete series of real groups were found *before* the representations to which they were associated were constructed (see [23, Theorem 16] and [47, Theorem 4]); whereas, in the p-adic setting, although we now have explicit constructions of many representations (see [1, 11–14, 26, 33–35, 50, 56], among many others), explicit character tables are still very rare.

This scarcity is of particular concern because, as suggested by Sally, it should be the case that "characters tell all" [46, p. 104]. Note, for example, the recent work of Langlands [31], which uses in a crucial way (see §1.d *loc. cit.*) the character formulæ of [43] to show the existence, but only for SL_2, of a transfer map dual to the transfer of stable characters. It seems likely that one of the main obstacles to extending the results of [31] to other groups is the absence of explicit character formulæ for them.

The good news here is that much *is* known about the behaviour of characters in general. For example, the Harish-Chandra–Howe local character expansion [18, 25, 27] and Murnaghan–Kirillov theory [28, 29, 37–41] give information about the asymptotics (near the identity element) of characters of p-adic groups in terms of Fourier transforms of orbital integrals (nilpotent or semisimple) on the Lie algebra, and many existing character formulæ are stated in terms of such orbital integrals (see, for example, [16, Theorem 5.3.2], [49, Theorems 6.6 and 7.18], [19, Lemma 10.0.4], and [4, Theorem 7.1]). See also [4, §0.1] for a more exhaustive description of what is known in the supercuspidal case.

The bad news is that many applications require completely explicit character tables—in particular, the evaluation of Fourier transforms of orbital integrals when they appear—but that Hales [22] has shown that the orbital integrals may themselves be 'non-elementary'. This term has a technical meaning, but, for our purposes, it suffices to regard it informally as meaning 'difficult to evaluate'. (Note, though, that the asymptotic behaviour of orbital integrals 'near ∞' is understood in all cases; see [55, Proposition VIII.1].) Since SL_2 is both simple enough for many explicit computations to be tractable (for example, the Fourier transforms of nilpotent orbital integrals have already been computed, in [17, Appendix A.3–A.4]), and complicated enough for interesting phenomena to be apparent (for example, unlike GL_2 and PGL_2, it admits non-stable characters), it is a natural focus for our investigations.

Another perspective on the behaviour of characters in the range where Murnaghan–Kirillov theory holds is offered in [15, Theorem 4.2(d)], [51, Proposition 2.9(2)], and [52, Theorem 2.5], where explicit mention of orbital integrals is replaced (on the 'bad shell'—see §10.2) by arithmetically interesting sums, identified in [51, 52] as Kloosterman sums. In fact, exponential sums—specifically, Gauss sums—have long been observed in p-adic harmonic analysis; see, for example, [48, §1.3], [55, §VIII.1], [16, p. 55], [15, Proposition 3.7], and [4, §5.2].

The work recorded here was carried out while preparing [5], which provides a proof of the aforementioned SL_2 character formulæ [43] by specialising the results

of [4, 19]. As discussed above, these general results are stated in terms of Fourier transforms of orbital integrals (see Definition 5.5); so, in order to obtain completely explicit formulæ, it was necessary to evaluate those Fourier transforms. The author of the present paper was surprised to discover that this latter evaluation reduced to the computation of *Bessel functions* (see §7 and Proposition 8.11); but, in retrospect, by the aforementioned p-adic Lefschetz principle, it seems natural that the 'special functions' described in [42] will play some important role in p-adic harmonic analysis, since their classical analogues are so integral to real harmonic analysis (see, for just one example, [21, Theorem 2], where Harish-Chandra's **c**-function is calculated in terms of Γ-functions). Relationships between a different sort of Bessel function, and a different sort of orbital integral (adapted to the Jacquet–Ye relative trace formula), have already been demonstrated by Baruch [6–10]. We will investigate further applications of complex-valued p-adic special functions in future work.

1.2. **Outline of the paper.** In order that everything be completely explicit, we need to carry around a large amount of notation; we describe it in §§2–7. Specifically, §2–4 describe the basic notation for working with groups over p-adic fields, adapted to the particular setting of the group SL_2. Since our formulæ will be written 'torus-by-torus' (*a la* Theorem 12 of [24]), we need to describe the tori in SL_2. This can be done very concretely; see Definition 4.1.

In §5, we define the functions $\hat{\mu}^G_{X^*}$ (Fourier transforms of orbital integrals) that we want to compute as representing functions for certain invariant distributions on $\mathfrak{sl}_2$ (see Definition 5.5 and Notation 5.7). Since these functions are defined only up to scalar multiples, it is important to be aware of the normalisations involved in their construction. In this respect, note that we specify the (Haar) measures that we are using in Definition 2.1 and Proposition 11.2.

As mentioned in §1.1, p-adic harmonic analysis tends to involve Gauss sums and other fourth roots of unity, and our calculations are no exception; we define and compare some of the relevant constants in §6. Finally, with these ingredients in place, we can follow [42] in defining the Bessel functions that we will use to evaluate $\hat{\mu}^G_{X^*}$. Already, [42] offers considerable information about the values of these functions, but we need to carry the calculations further, especially far from the identity (see Proposition 7.5) and on the 'bad shell' (see Proposition 7.7)—where (twisted) Kloosterman sums make an appearance.

In §8, we define a function $M^G_{X^*}$ (see Definition 8.4), which we will spend most of the rest of the paper computing. This is a reasonable focus because, once the computations are completed, Proposition 11.2 will show that we have actually been computing $\hat{\mu}^G_{X^*}$. The definition of $M^G_{X^*}$ involves a rather remarkable function φ_θ (see Definition 8.2 and Lemma 8.3); it seems likely that generalising our techniques will require understanding the proper replacement for φ_θ.

Proposition 8.11 describes $M^G_{X^*}$ in terms of Bessel functions, and Proposition 8.13 uses Theorem 7.4 to describe their behaviour near 0.

We now proceed according to the 'type' of X^* (as in Definition 4.4). The calculations when X^* is split, and when it is unramified, are quite similar; we combine them in §9. We split into cases depending on whether the argument to $M^G_{X^*}$ is far from (as in §9.1) or close to (as in §9.2) zero; there are qualitative differences in the behaviour, as can be seen by comparing, for example, Theorems 9.5 and 9.7. When X^* is ramified, it turns out that, in addition to the behaviour far from

(as in §10.1) and close to (as in §10.3) zero, there is a third range of interest in the middle. This is the so called 'bad shell' (see §10.2), and it seems likely that the particularly complicated nature of the formulæ here is a reflection of the 'non-elementary' behaviour of orbital integrals (hence, by Murnaghan–Kirillov theory, also of characters) described in [22].

Finally, we show in §11 that the function that we have been evaluating actually does represent the desired distribution, i.e., is equal to $\hat{\mu}^G_{X^*}$. (See Proposition 11.2.) We close with some observations (see Theorem 11.3) about the qualitative behaviour of orbital integrals that does not depend (much) on the 'type' of X^*.

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2. Notation

Suppose that k is a non-discrete, non-Archimedean local field. We do not make any assumptions on its characteristic, but we assume that its residual characteristic p is not 2. (We occasionally cite [48], which works only with characteristic-0 fields; but we shall not use any results from there that require this restriction.) Let R denote the ring of integers in k, $\wp$ the prime ideal of R, and ord the valuation on k with value group $\mathbb{Z}$.

Let $\mathfrak{f}$ denote the residue field $R/\wp$ of k. We write $q = |\mathfrak{f}|$ for the number of elements in $\mathfrak{f}$, and put $\left|x\right| = q^{-\operatorname{ord}(x)}$ for $x \in k$. If $\alpha \in \mathbb{C}$, then we will write ν^α for the (multiplicative) character $x \mapsto \left|x\right|^\alpha$ of $k^\times$.

Put $\mathbf{G} = \mathrm{SL}_2$ and $G = \mathbf{G}(k)$, and let $\mathfrak{g}$ and $\mathfrak{g}^*$ denote the Lie algebra and dual Lie algebra of G, respectively.

It is important for our calculations to be quite specific about the Haar measures that we are using. For convenience, we fix the ones used in [42] (see p. 280 *loc. cit.*).

Definition 2.1. Throughout, we shall use the (additive) Haar measure $\mathrm{d}x$ on k that assigns measure 1 to R, and the associated (multiplicative) Haar measure $\mathrm{d}^\times x = \left|x\right|^{-1}\mathrm{d}x$ on $k^\times$ that assigns measure $1 - q^{-1}$ to $R^\times$. When convenient, we shall write $\mathrm{d}t$ instead of $\mathrm{d}x$.

Definition 2.2. If Φ is an (additive) character of k, then we define $\Phi_b : x \mapsto \Phi(bx)$ for $b \in k$. The *depth* of Φ is

$$\mathrm{d}(\Phi) := \min\left\{i \in \mathbb{Z} : \Phi \text{ is trivial on } \wp^{i+1}\right\}$$

(if Φ is non-trivial) and $\mathrm{d}(\Phi) = -\infty$ otherwise.

The depth of a character is related to what is often called its *conductor* by $\mathrm{d}(\Phi) = \omega(\Phi) - 1$ (in the notation of [48, §1.3]). We have that

$$\mathrm{d}(\Phi_b) = \mathrm{d}(\Phi) - \operatorname{ord}(b). \tag{2.3}$$

Note that the notion of depth, and the symbol d, will be used in multiple contexts (see Definition 4.9); we rely on the context to disambiguate them.

Notation 2.4. Φ is a non-trivial (additive) character of k.

One of the crucial tools of Harish-Chandra's approach to harmonic analysis is the reduction, whenever possible, of questions about a group to questions about its Lie algebra. The exponential map often allows one to effect this reduction, but, since it might converge only in a very small neighbourhood of 0, we replace it with a 'mock-exponential map' (see [1, §1.5]) which has many of the same properties (see Lemma 2.6).

Definition 2.5. The *Cayley map* $\mathrm{c} : k \setminus \{1\} \to k \setminus \{-1\}$ is defined by

$$\mathrm{c}(X) = (1 + X)(1 - X)^{-1} \quad \text{for } X \in k \setminus \{1\}.$$

The Cayley function is available in many settings; note that we are using it only as a function defined almost everywhere on k. We gather a few of its properties below.

Lemma 2.6.

- *The map* c *is a bijection.*
- $\mathrm{c}(-X) = \mathrm{c}(X)^{-1} = \mathrm{c}^{-1}(X)$ *for* $X \in k \setminus \{\pm 1\}$.
- *The map* c *carries* $\wp^i$ *to* $1 + \wp^i$ *for all* $i \in \mathbb{Z}_{>0}$.
- *In the notation of Definition 2.1, the pull-back along* c *of the measure* $\mathrm{d}^\times x$ *on* $1 + \wp$ *is the measure* $\mathrm{d}x$ *on* $\wp$.
- *If* $X \in \wp^i$ *and* $Y \in \wp^j$*, with* $i, j \in \mathbb{Z}_{>0}$*, then*

  $$\mathrm{c}(X + Y) \equiv \mathrm{c}(X) + 2Y \pmod{1 + \wp^n},$$

  where $n = j + \min\{2i, j\}$.

Proof. It is easy to check that $x \mapsto (1 - x)(1 + x)^{-1}$ is inverse to c and satisfies the desired equalities, and that $\mathrm{c}(\wp^i) \subseteq 1 + \wp^i$ and $\mathrm{c}^{-1}(1 + \wp^i) \subseteq \wp^i$. If $f \in C^\infty(1 + \wp)$, then there is some $i \in \mathbb{Z}_{>0}$ such that $f \in C(1 + \wp/1 + \wp^i)$. Upon noting that $\operatorname{meas}_{\mathrm{d}x}(\wp^i) = q^{-i} = \operatorname{meas}_{\mathrm{d}^\times x}(1 + \wp^i)$, we see that

$$\int_{1+\wp} f(x)\mathrm{d}^\times x = \sum_{x \in 1+\wp/1+\wp^i} f(x) \operatorname{meas}_{\mathrm{d}^\times x}(1 + \wp^i)$$
$$= \sum_{x \in \wp/\wp^i} (f \circ \mathrm{c})(x) q^{-i} \operatorname{meas}_{\mathrm{d}x}(\wp^i) = \int_\wp (f \circ \mathrm{c})(x)\mathrm{d}x.$$

Finally, under the stated conditions on X and Y,

$$\begin{aligned}\bigl(\mathrm{c}(X) + 2Y\bigr)\bigl(1 - (X + Y)\bigr) \\ &= \mathrm{c}(X) \cdot (1 - X) + Y\bigl(2(1 - (X + Y)) - \mathrm{c}(X)\bigr) \\ &= (1 + X + Y) + Y\bigl((1 - 2X - \mathrm{c}(X)) - 2Y\bigr).\end{aligned}$$

Since $\mathrm{c}(X) = 1 + 2X(1 - X)^{-1}$, we have that $1 - 2X - \mathrm{c}(X) \in \wp^{2i}$. The result follows. $\square$

3. Fields and algebras

Definition 3.1. For $\theta \in k^\times$, we write k_θ for the k-algebra that is $k \oplus k$ (as a vector space), equipped with multiplication $(a, b) \cdot (c, d) = (ac + bd\theta, ad + bc)$. We write $\sqrt{\theta}$ for the element $(0, 1) \in k_\theta$, so that $(a, b) = a + b\sqrt{\theta}$.

We also use the notation $\sqrt{\theta}$ for a matrix (see Definition 4.1); we shall rely on context to make the meaning clear.

If $\theta \notin (k^\times)^2$, then k_θ is isomorphic to $k(\sqrt{\theta})$ (as k-algebras) via the map $(a, b) \mapsto a + b\sqrt{\theta}$, and we shall not distinguish between them.

If $\theta = x^2$, with $x \in k$, then k_θ is isomorphic to $k \oplus k$ (as k-algebras) via the map $(a, b) \mapsto (a + bx, a - bx)$.

Definition 3.2. We define

$$N_\theta(a + b\sqrt{\theta}) = a^2 - b^2\theta, \quad \operatorname{tr}_\theta(a + b\sqrt{\theta}) = 2a,$$
$$\operatorname{Re}_\theta(a + b\sqrt{\theta}) = a, \quad \operatorname{Im}_\theta(a + b\sqrt{\theta}) = b,$$

and

$$\operatorname{ord}_\theta(a + b\sqrt{\theta}) = \tfrac{1}{2}\operatorname{ord}\bigl(N_\theta(a + b\sqrt{\theta})\bigr)$$

for $a + b\sqrt{\theta} \in k_\theta$. Write $C_\theta = \ker N_\theta$ and $V_\theta = \ker \operatorname{tr}_\theta$, and let $\operatorname{sgn}_\theta$ be the unique (multiplicative) character of $k^\times$ with kernel precisely $N_\theta(k_\theta^\times)$.

If $\theta \notin (k^\times)^2$, then N_θ and tr_θ are the usual norm and trace maps associated to the quadratic extension of fields k_θ/k, and $\operatorname{ord}_\theta$ is the valuation on k_θ extending ord. In any case, $k_\theta^\times = \bigl\{z \in k_\theta \,:\, N_\theta(z) \neq 0\bigr\}$.

We can describe the signum character explicitly by

$$\operatorname{sgn}_\theta(x) = \begin{cases} 1, & \theta \text{ split} \\ (-1)^{\operatorname{ord}(x)}, & \theta \text{ unramified}, \end{cases} \tag{3.3}$$

and

$$\begin{aligned} \operatorname{sgn}_\theta(\theta) &= \operatorname{sgn}_{\mathfrak{f}}(-1) \\ \operatorname{sgn}_\theta(x) &= \operatorname{sgn}_{\mathfrak{f}}(\overline{x}) \quad \text{for } x \in R^\times, \end{aligned} \tag{3.4}$$

where $\operatorname{sgn}_{\mathfrak{f}}$ is the quadratic character of $\mathfrak{f}^\times$ and $x \mapsto \overline{x}$ the reduction map $R \to \mathfrak{f}$.

4. Tori and filtrations

We begin by defining a few model tori.

Definition 4.1. For $\theta \in k$, put

$$\mathbf{T}_\theta = \left\{\begin{pmatrix} a & b \\ b\theta & a \end{pmatrix} \,:\, a^2 - b^2\theta = 1\right\}.$$

Then

$$\mathfrak{t}_\theta := \operatorname{Lie}(\mathbf{T}_\theta) = \left\{\begin{pmatrix} 0 & b \\ b\theta & 0 \end{pmatrix}\right\}.$$

We write $\sqrt{\theta}$ for the element $\left(\begin{smallmatrix} 0 & 1 \\ \theta & 0 \end{smallmatrix}\right) \in \mathfrak{t}_\theta$, so that $\mathfrak{t}_\theta = \operatorname{Span}_k \sqrt{\theta}$. We will call a maximal k-torus in $\mathbf{G}$ *standard* exactly when it is of the form $\mathbf{T}_\theta$ for some $\theta \in k$.

We also use the notation $\sqrt{\theta}$ for an element of an extension of k (see Definition 4.1); we shall rely on context to make the meaning clear.

Remark 4.2. The group T_θ is isomorphic to $C_\theta = \ker N_\theta$, and the Lie algebra $\mathfrak{t}_\theta$ to $V_\theta = \ker \operatorname{tr}_\theta$, in each case via the map $\left(\begin{smallmatrix} a & b \\ b\theta & a \end{smallmatrix}\right) \mapsto (a, b)$.

We shall use the terms ‘split’, ‘unramified’, and ‘ramified’ in many different contexts.

Remark 4.3. If $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$ and $\mathfrak{t} = \mathrm{Lie}(T)$, then we shall identify $\mathfrak{t}$ (respectively, $\mathfrak{t}^*$) with the spaces of fixed points for the adjoint (respectively, co-adjoint) action on $\mathfrak{g}$ (respectively, $\mathfrak{g}^*$). By abuse of language, we shall sometimes say that $X^* \in \mathfrak{g}^*$ or $Y \in \mathfrak{g}$ lies in, or belongs to, the torus $\mathbf{T}$ to mean that $X^* \in \mathfrak{t}^*$ and $Y \in \mathfrak{t}$; equivalently, that $C_{\mathbf{G}}(X^*) = \mathbf{T} = C_{\mathbf{G}}(Y)$. In particular, "$X^*$ and Y belong to a common torus" is shorthand for "$C_{\mathbf{G}}(X^*) = C_{\mathbf{G}}(Y)$".

Definition 4.4. A maximal k-torus in $\mathbf{G}$ is called (un)ramified according as it is elliptic and splits over an (un)ramified extension of k. An element $\theta \in k$ is called split, unramified, or ramified according as $\mathbf{T}_\theta$ has that property. A regular, semisimple element of $\mathfrak{g}$ or $\mathfrak{g}^*$ is called split, unramified, or ramified according as the torus to which it belongs has that property.

Remark 4.5. To be explicit, squares in $k^\times$ are split, and a non-square $\theta \in k$ is unramified or ramified according as $\max\left\{\mathrm{ord}(x^2\theta) \,:\, x \in k\right\}$ is even or odd, respectively.

Notation 4.6. If $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$, with $T = \mathbf{T}(k)$, then we write $W(\mathbf{G}, \mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ for the absolute, and $W(G, T) = N_G(T)/T$ for the relative, Weyl group of $\mathbf{T}$ in $\mathbf{G}$.

Every maximal k-torus in $\mathbf{G}$ is G-conjugate to some $\mathbf{T}_\theta$. (See, for example, [17, §A.2].) In particular,

$$\mathrm{Int}\begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}\left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,:\, ad = 1\right\} = \mathbf{T}_1.$$

Remark 4.7. For all $\theta \in k$, the group $W(\mathbf{G}, \mathbf{T}_\theta)$ has order 2, with the non-trivial element acting on $\mathbf{T}_\theta$ by inversion. If $\mathrm{sgn}_\theta(-1) = 1$ (in particular, if θ is split or unramified), say, with $N_\theta(a + b\sqrt{\theta}) = -1$, then $W(G, T_\theta)$ also has order 2, with the non-trivial element represented by $\left(\begin{smallmatrix} a & b \\ -b\theta & -a \end{smallmatrix}\right)$. If $\theta = 1$, then we may take $(a, b) = (0, 1)$ to recover the familiar Weyl-group element. Otherwise (i.e., if $\mathrm{sgn}_\theta(-1) = -1$), $W(G, T_\theta)$ is trivial.

The concept of *stable conjugacy* was introduced by Langlands as part of the foundation of the Langlands conjectures; see [30, pp. 2–3].

Definition 4.8. Two

- maximal k-tori $\mathbf{T}_i$ in $\mathbf{G}$,
- regular semisimple elements $X_i^* \in \mathfrak{g}^*$, or
- regular semisimple elements $Y_i \in \mathfrak{g}$,

with $i = 1, 2$, are called *stably conjugate* exactly when there are a field extension E/k and an element $g \in \mathbf{G}(E)$ such that

- $\mathrm{Int}(g)T_1 = T_2$ or
- $\mathrm{Ad}^*(g)X_1^* = X_2^*$ or
- $\mathrm{Ad}(g)X_1 = X_2$,

where $T_i = \mathbf{T}_i(k)$ for $i = 1, 2$. If the conjugacy can be carried out without passing to an extension field (i.e., if we may take $g \in G$), then we will sometimes emphasise this by saying that the tori or elements are *rationally conjugate.*

Note that the Zariski-density of T_i in $\mathbf{T}_i$ implies that $\mathrm{Int}(g)\mathbf{T}_1 = \mathbf{T}_2$, but that this is a strictly weaker condition; indeed, given *any* two maximal tori, there is an element g, defined over some extension field of k, satisfying this condition. In our

special case (of $\mathbf{G} = \mathrm{SL}_2$), we have that two tori or elements are stably conjugate if and only if they are conjugate in $\mathrm{GL}_2(k)$.

More concretely, two tori $\mathbf{T}_\theta$ and $\mathbf{T}_{\theta'}$ are stably conjugate if and only if $\theta \equiv \theta' \pmod{(k^\times)^2}$. The stable conjugacy class of the split torus $\mathbf{T}_1$ is also a rational conjugacy class.

Suppose that ϵ is an unramified, and ϖ a ramified, non-square. Then the stable conjugacy class of $\mathbf{T}_\epsilon$ splits into 2 rational conjugacy classes, represented by $\mathbf{T}_\epsilon$ and $\mathbf{T}_{\varpi^2\epsilon}$. The stable conjugacy class of $\mathbf{T}_\varpi$ is also a rational conjugacy class if $\mathrm{sgn}_\varpi(-1) = -1$; but it splits into 2 rational conjugacy classes, represented by $\mathbf{T}_\varpi$ and $\mathbf{T}_{\epsilon^2\varpi}$, if $\mathrm{sgn}_\varpi(-1) = 1$.

We also need filtrations on the Lie algebra, and dual Lie algebra, of a torus. These definitions are standard (see, for example, [1, §1.4]) and can be made in far more generality (see [35, §3] and [36, §3.3]); we give only simple definitions adapted to $\mathbf{G} = \mathrm{SL}_2$.

Definition 4.9. Let $\mathbf{T}$ be a maximal k-torus in $\mathbf{G}$, and put $\mathfrak{t} = \mathrm{Lie}(\mathbf{T}(k))$. Recall that $\mathbf{T}$ is G-conjugate to $\mathbf{T}_\theta$ for some $\theta \in k$, so that $\mathfrak{t} = \mathrm{Lie}(T)$ is isomorphic to $V_\theta = \ker \mathrm{tr}_\theta \subseteq k_\theta$. For $r \in \mathbb{R}$, we write $\mathfrak{t}_r$ for the pre-image of $\{Y \in V_\theta : \mathrm{ord}_\theta(Y) \ge r\}$ and $\mathfrak{t}_{r+}$ for the pre-image of $\{Y \in V_\theta : \mathrm{ord}_\theta(Y) > r\}$; and then we write $\mathfrak{t}^*_r = \{X^* \in \mathfrak{t}^* : \Phi(\langle X^*, Y\rangle) = 1 \text{ for all } Y \in \mathfrak{t}_{(-r)+}\}$ (where Φ is the additive character of Notation 2.4).

If $X^* \in \mathfrak{t}^*$ and $Y \in \mathfrak{t}$, then we define $\mathrm{d}(X^*) = \max\{r \in \mathbb{R} : X^* \in \mathfrak{t}^*_r\}$ and $\mathrm{d}(Y) = \max\{r \in \mathbb{R} : Y \in \mathfrak{t}_r\}$.

One can define a notion of depth in more generality (see, for example, [2, §3.3 and Example 3.4.6] and [28, §2.1 and Lemma 2.1.5]), but we only need the special case above. (The only remaining case to consider for $\mathfrak{g} = \mathfrak{sl}_2(k)$ is the depth of a nilpotent element, which is ∞.)

5. Orbital integrals

Our goal in this paper is to compute Fourier transforms of regular, semi-simple orbital integrals on $\mathfrak{g}$ (see Definition 5.5 below). Since the Fourier transforms of nilpotent orbital integrals were computed in [17, Appendix A], this covers all Fourier transforms of orbital integrals on $\mathfrak{g}$ (for our particular case $\mathbf{G} = \mathrm{SL}_2$). The case of orbital integrals on G was discussed in [45], as the culmination of the series of papers that began with [43, 44].

We will begin by choosing a representative for the regular, semi-simple orbit of interest. By §4, we may choose this representative in a standard torus (in the sense of Definition 4.1).

Notation 5.1. $\beta, \theta \in k^\times$, and $X^* = \beta \cdot \sqrt{\theta} \in \mathfrak{t}^*_\theta$.

Here, we are implicitly using the identification of $\mathfrak{t}_\theta$ with $\mathfrak{t}^*_\theta$ via the trace form; what we really mean is that $\langle X^*, Y\rangle = \mathrm{tr}\,\beta \cdot \sqrt{\theta} \cdot Y$ for $Y \in \mathfrak{t}_\theta$, where $\langle\cdot,\cdot\rangle$ is the usual pairing between $\mathfrak{t}^*_\theta$ and $\mathfrak{t}_\theta$.

As in Definition 2.2, we may define a new character Φ_β of k. This character will occur often enough in our calculations that it is worthwhile to give it a name.

Notation 5.2. $-r = \mathrm{d}(X^*)$, $\Phi' = \Phi_\beta$, and $r' = \mathrm{d}(\Phi')$.

By Definition 4.9, $Y \mapsto \Phi(\langle X^*, Y\rangle)$ is trivial on $(\mathfrak{t}_\theta)_{r+}$, but not on $(\mathfrak{t}_\theta)_r$. Therefore, $r' = r + \frac{1}{2}\operatorname{ord}(\theta)$.

Since $C_G(X^*) = T_\theta$ is Abelian, it is unimodular; so there exists a measure on $G/C_G(X^*)$ invariant under the action of G by left translation.

Notation 5.3. Let $\mathrm{d}\dot{g}$ be a translation-invariant measure on $G/C_G(X^*)$.

Since the orbit, $\mathcal{O}^G_{X^*}$, of X^* under the co-adjoint action of G is isomorphic as a G-set to $G/C_G(X^*)$, we could transport to it the measure on the latter space; but we do not find it convenient to do so.

Since X^* is semisimple, $\mathcal{O}^G_{X^*}$ is closed in $\mathfrak{g}^*$ (see, for example, Proposition 34.3.2 of [53]). Therefore, the restriction to $\mathcal{O}^G_{X^*}$ of a locally constant, compactly supported function on $\mathfrak{g}^*$ remains locally constant and compactly supported, so that the following definition makes sense.

Definition 5.4. The orbital integral of X^* is the distribution $\mu^G_{X^*}$ on $\mathfrak{g}^*$ defined by

$$\mu^G_{X^*}(f^*) = \int_{G/C_G(X^*)} f^*(\mathrm{Ad}^*(g)X^*)\mathrm{d}\dot{g} \quad \text{for all } f^* \in C_c^\infty(\mathfrak{g}^*).$$

We are interested in the Fourier transform of $\mu^G_{X^*}$. The definition of the Fourier transform (of distributions or of functions) requires, in addition to a choice of additive character (see Notation 2.4), also a choice of Haar measure $\mathrm{d}Y$ on $\mathfrak{g}^*$; but we shall build this choice into our representing function (see Notation 5.7), so that it will not show up in our final answer.

Definition 5.5. The Fourier transform of the orbital integral of X^* is the distribution $\hat{\mu}^G_{X^*}$ on $\mathfrak{g}$ defined for all $f \in C_c^\infty(\mathfrak{g})$ by

$$\hat{\mu}^G_{X^*}(f) = \mu^G_{X^*}(\hat{f}),$$

where

$$\hat{f}(Y^*) = \int_{\mathfrak{g}} f(Y)\Phi(\langle Y^*, Y\rangle)\mathrm{d}Y \quad \text{for all } Y^* \in \mathfrak{g}^*.$$

It is a result of Harish-Chandra (see [25, Theorem 1.1]) that $\hat{\mu}^G_{X^*}$ is *representable* on $\mathfrak{g}$; i.e., that there exists a locally integrable function F on $\mathfrak{g}$ such that

$$\hat{\mu}^G_{X^*}(f) = \int_G f(Y)F(Y)\mathrm{d}Y \quad \text{for all } f \in C_c^\infty(\mathfrak{g}).$$

One can say more about the behaviour and asymptotics of the function F. For example, it turns out that it blows up as Y approaches 0, but that its blow-up is controlled by a power of a discriminant function.

Definition 5.6. The *Weyl discriminant* on $\mathfrak{g}$ is the function $D_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{C}$ such that, for all $Y \in \mathfrak{g}$, $D_{\mathfrak{g}}(Y)$ is the coefficient of the degree-1 term in the characteristic polynomial of $\mathrm{ad}(Y)$. Concretely,

$$D_{\mathfrak{g}}\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = 4(a^2 + bc).$$

Our main interest, however, is in the restriction of the function F above to the set $\mathfrak{g}^{\mathrm{rss}}$ of regular, semisimple elements, where it is locally constant.

Notation 5.7. By abuse of notation, we write again $\hat{\mu}^G_{X^*}$ for the function that represents the restriction to $\mathfrak{g}^{\mathrm{rss}}$ of $\hat{\mu}^G_{X^*}$.

When we refer to the computation of the Fourier transform of an orbital integral, it is actually the (scalar) function of Notation 5.7 that we are trying to compute. The main tool in this direction is a general integral formula of Huntsinger (see [3, Theorem A.1.2]), but we find it easier to evaluate an integral adapted to our current setting (see Definition 8.4). The computation of this integral will occupy most of the paper; once that is done, we shall finally prove that it actually represents the distribution $\hat\mu^G_{X^*}$ (see Proposition 11.2).

Finally, we fix an element at which to evaluate the functions of interest. Since $\hat\mu^G_{X^*}$, as just defined, and $M^G_{X^*}$ below (see Definition 8.4) are G-invariant functions on $\mathfrak{g}^{\mathrm{rss}}$, we may again consider only elements of standard tori.

Notation 5.8. $s, \theta' \in k^\times$, and $Y = s \cdot \sqrt{\theta'} \in \mathfrak{t}_{\theta'}$.

Our computations will be phrased in terms of the values of two 'basic' functions at Y.

Lemma 5.9. $\mathrm{d}(Y) = \frac{1}{2}\operatorname{ord}(s^2\theta')$ *and* $D_{\mathfrak{g}}(Y) = 4s^2\theta'$.

Proof. This is a straightforward consequence of Definitions 4.9 and 5.6. □

6. Roots of unity and other constants

The computation of Fourier transforms of orbital integrals on $\mathfrak{g}$—hence, via Murnaghan–Kirillov theory [3, 4, 28, 29, 38], also of the values near the identity of characters of G (cf. [5, 43])—involves a somewhat bewildering array of 4th roots of unity, for each of which there is a variety of notation available. It turns out that all of these can be expressed in terms of a single 'basic' quantity, the Gauss sum, denoted by $G(\Phi)$ in [48, Lemma 1.3.2]. The definition there implicitly depends on a choice of uniformiser, denoted there by π. Although the choice is arbitrary, we shall find it convenient for later usage to denote it by $-\varpi$. Recall from Notation 2.4 that Φ is a non-trivial (additive) character of k.

Definition 6.1. If ϖ is a uniformiser of k, then

$$G_\varpi(\Phi) := q^{-1/2} \sum_{X \in R/\wp} \Phi_{(-\varpi)^{\mathrm{d}(\Phi)}}(X^2).$$

It is possible to compute these values exactly (see, for example, [32, Theorem 5.15]), but we shall only require a few transformation laws.

Lemma 6.2. *If ϖ is a uniformiser of k, then*

$$\begin{aligned} G_{b\varpi}(\Phi) &= \operatorname{sgn}_\varpi(b)^{\mathrm{d}(\Phi)} G_\varpi(\Phi) \quad \textit{for } b \in R^\times, \\ G_\varpi(\Phi_b) &= \operatorname{sgn}_\varpi(b) G_\varpi(\Phi) \quad \textit{for } b \in k^\times, \\ G_\varpi(\Phi)^2 &= \operatorname{sgn}_\varpi(-1), \end{aligned}$$

and

$$G_\varpi(\Phi) = q^{-1/2} \operatorname{sgn}_\varpi(-1)^{\mathrm{d}(\Phi)} \sum_{X \in \mathfrak{f}^\times} \overline{\Phi}(X) \operatorname{sgn}_{\mathfrak{f}}(X),$$

where $\operatorname{sgn}_{\mathfrak{f}}$ *is the quadratic character of* $\mathfrak{f}^\times$*, and* $\overline{\Phi}$ *the (additive) character of* $\mathfrak{f} = R/\wp$ *arising from the restriction to* R *of the depth-*0 *character* $\Phi_{\varpi^{\mathrm{d}(\Phi)}}$ *of* k*.*

Proof. Since $\sum_{X\in\mathfrak{f}} \overline{\Phi}(X) = 0$, we have that

$$\begin{aligned}\sum_{X\in\mathfrak{f}^\times} \overline{\Phi}(X)\operatorname{sgn}_{\mathfrak{f}}(X) &= \overline{\Phi}(0) + \sum_{X\in\mathfrak{f}^\times} \overline{\Phi}(X)\bigl(1+\operatorname{sgn}_{\mathfrak{f}}(X)\bigr) \\ &= \overline{\Phi}(0) + 2\sum_{X\in(\mathfrak{f}^\times)^2} \overline{\Phi}(X) \\ &= \sum_{X\in\mathfrak{f}} \overline{\Phi}(X^2) \\ &= q^{1/2} G_\varpi(\Phi_{(-1)^{\mathrm{d}(\Phi)}}).\end{aligned}$$

In other words,

$$(*) \qquad G_\varpi(\Phi_{(-1)^{\mathrm{d}(\Phi)}}) = q^{-1/2} G(\operatorname{sgn}_{\mathfrak{f}}, \overline{\Phi}),$$

where the notation on the right is as in [32, §5.2] (except that their ψ is our $\operatorname{sgn}_{\mathfrak{f}}$, the quadratic character of $\mathfrak{f}^\times$, and their χ is our $\overline{\Phi}$). The third equality, and the second equality for $b \in R^\times$, now follow from Theorem 5.12 *loc. cit.* The first equality follows from the second upon noting that $G_{b\varpi}(\Phi) = G_\varpi(\Phi_{b^{\mathrm{d}(\Phi)}})$; and taking $b = (-1)^{\mathrm{d}(\Phi)}$ and combining with $(*)$ gives the fourth equality. Finally, by definition, $G_\varpi(\Phi_{(-\varpi)^n}) = G_\varpi(\Phi) = \operatorname{sgn}_\varpi(-\varpi)^n G_\varpi(\Phi)$ for all $n \in \mathbb{Z}$. □

By Proposition 8.11 and Theorem 7.4, our calculations will involve the Γ-factors defined in [42, §3]. Of particular interest is $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)$. By Theorem 3.1(iii) *loc. cit.*, $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)^2 = \operatorname{sgn}_\varpi(-1)$, so that, by Lemma 6.2, $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) = \pm G_\varpi(\Phi)$. It will be useful to identify the sign.

Lemma 6.3. *If ϖ is a uniformiser of k, then* $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) = \operatorname{sgn}_\varpi(-1)^{\mathrm{d}(\Phi)+1} G_\varpi(\Phi)$.

Proof. Write $\overline{\Phi} = \Phi_{\varpi^{\mathrm{d}(\Phi)}}$; this is a depth-0 character of k. The definitions of [42] depend on a depth-(-1) additive character χ; we take it to be $\overline{\Phi}_\varpi$. The definition of $\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)$ involves a principal-value integral (see Definition 8.4), but, as pointed out in the proof of [42, Theorem 3.1], we have by Lemma 3.1 *loc. cit.* and (3.4) that it simplifies to

$$\begin{aligned}\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) &= \int_{\operatorname{ord}(x)=-1} \overline{\Phi}_\varpi(x)\bigl|x\bigr|^{1/2}\operatorname{sgn}_\varpi(x)\mathrm{d}^\times x \\ &= \int_{R^\times} \overline{\Phi}_\varpi(\varpi^{-1}x)\bigl|\varpi^{-1}x\bigr|^{1/2}\operatorname{sgn}_\varpi(\varpi^{-1}x)\mathrm{d}^\times x \\ &= q^{1/2}\operatorname{sgn}_\varpi(-1)\operatorname{meas}_{\mathrm{d}^\times x}(1+\wp)\sum_{x\in R^\times/1+\wp} \overline{\Phi}(x)\operatorname{sgn}_{\mathfrak{f}}(x),\end{aligned}$$

where $\mathrm{d}^\times x$ is the Haar measure on $k^\times$ with respect to which $\operatorname{meas}_{\mathrm{d}^\times x}(R^\times) = 1-q^{-1}$ (see Definition 2.1). Since $\operatorname{meas}_{\mathrm{d}^\times x}(1+\wp) = q^{-1}$, the result now follows from Lemma 6.2. □

We will also need some constants associated to specific elements.

In [55, Proposition VIII.1], Waldspurger describes the ‘behaviour at ∞’ of Fourier transforms of semisimple orbital integrals on general reductive, p-adic Lie algebras. His description involves a 4th root of unity $\gamma_\psi(X^*, Y)$ (cf. p. 79 *loc. cit.*); since his ψ is our Φ (see Notation 2.4), we denote it by $\gamma_\Phi(X^*, Y)$. See Theorem 11.3 for our quantitative analogues (for the special case of $\mathfrak{sl}_2$) of his result.

Although we would like to do so (see Remark 6.9), it is notationally unwieldy to avoid any longer choosing 'standard' representatives for $k^\times/(k^\times)^2$. Although our proofs will make use of these choices, none of the statements of the main results (except Theorems 10.8 and 10.9, via Remark 10.7) rely on them.

Notation 6.4. Let ϵ be a lift to $R^\times$ of a non-square in $\mathfrak{f}^\times$, and ϖ a uniformiser of k.

Definition 6.5. Recall Notations 5.1 and 5.8. If X^* and Y lie in stably conjugate tori, so that $\theta \equiv \theta' \pmod{(k^\times)^2}$, then

$$\gamma_\Phi(X^*, Y) = \begin{cases} 1, & \theta \equiv 1 \\ \gamma_{\text{un}}(s), & \theta \equiv \epsilon \\ \gamma_{\text{ram}}(s), & \theta \equiv \varpi \\ -\gamma_{\text{un}}(s)\gamma_{\text{ram}}(s), & \theta \equiv \epsilon\varpi, \end{cases}$$

where all congruences are taken modulo $(k^\times)^2$, and where

$$\gamma_{\text{un}}(s) := (-1)^{r'+1}\operatorname{sgn}_\epsilon(s) \quad\text{and}\quad \gamma_{\text{ram}}(s) := \operatorname{sgn}_\varpi(-s)G_\varpi(\Phi')$$

(with notation as in Notation 5.2 and Definition 6.1). It simplifies our notation considerably also to put $\gamma_\Phi(X^*, Y) = 1$ if X^* is elliptic and Y is split, and otherwise put $\gamma_\Phi(X^*, Y) = 0$ if X^* and Y do not lie in stably conjugate tori.

Remark 6.6. The dependence of $\gamma_\Phi(X^*, Y)$ on X^* is via r' and Φ' (see Notation 5.2). Expanding these definitions shows that $\gamma_\Phi(X^*, Y) = c_{\theta,\phi}\cdot\operatorname{sgn}_\theta(\beta s)$ when X^* and Y lie in stably conjugate tori, where the notation is as in Notations 5.1 and 5.8.

Notice that we have defined $\gamma_\Phi(X^*, Y)$ only when X^* and Y belong to (possibly different) standard tori, in the sense of Definition 4.1. A direct computation shows that, if we replace X^* or Y by a rational conjugate, or replace the pair (X^*, Y) by a stable conjugate, such that X^* and Y still lie in standard tori, then the constant $\gamma_\Phi(X^*, Y)$ does not change. (In the notation of Definition 8.2, $\operatorname{Ad}^*(g)X^*$ lies in a standard torus if and only if $\varphi_\theta(g) = (\alpha, 0)$, in which case $\operatorname{Ad}^*(g)X^* = \beta N_\theta(\alpha)\cdot\sqrt{N_\theta(\alpha)^{-2}\theta}$; and similarly for Y.) This allows us to define $\gamma_\Phi(X^*, Y)$ for all pairs of regular, semisimple elements, if desired.

By Lemma 6.2,

$$\gamma_{\text{ram}}(s)^2 = \operatorname{sgn}_\varpi(-1). \tag{6.7}$$

In order to make use of Propositions 7.5 and 7.7 below, we will need the computation

$$\begin{aligned}\operatorname{sgn}_\varpi(v)G_\varpi(\Phi'_{\varpi^{r'+1}}) \\ = \operatorname{sgn}_\varpi(\varpi^{-(r'+1)}s\theta)\cdot\operatorname{sgn}_\varpi(\varpi^{r'+1})G_\varpi(\Phi') = \operatorname{sgn}_\varpi(-\theta)\gamma_{\text{ram}}(s).\end{aligned} \tag{6.8}$$

Remark 6.9. We will be interested exclusively in the case when $\theta \in \{1, \epsilon, \varpi\}$. This means that we seem to be omitting the cases when $\theta \in \left\{\varpi^2\epsilon, \epsilon^2\varpi, \epsilon^{\pm 1}\varpi\right\}$; but, actually, this problem is not serious. Indeed, for $b \in k$, write $g_b := \left(\begin{smallmatrix}1 & 0\\ 0 & b\end{smallmatrix}\right) \in \operatorname{GL}_2(k)$. Then

$$\operatorname{Ad}^*(g_b)X^* = \operatorname{Ad}^*(g_b)(\beta\cdot\sqrt{\theta}) = \beta b^{-1}\cdot\sqrt{b^2\theta}$$

(where we identify $\mathfrak{t}_\theta^*$ with $\mathfrak{t}_\theta$ via the trace pairing, as in Notation 5.1); and $\hat\mu^G_{X^*} = \hat\mu^G_{\operatorname{Ad}^*(g_b)X^*}\circ\operatorname{Ad}(g_b)$. This covers $\theta = \varpi^2\epsilon$ (by taking $b = \varpi^{-1}$) and $\theta = \epsilon^2\varpi$ (by

taking $b = \epsilon^{-1}$). Handling $\theta \in \left\{\epsilon^{\pm 1}\varpi\right\}$ requires a different observation: since our choice of uniformiser was arbitrary, it could as well have been $\epsilon^{\pm 1}\varpi$ (or, for that matter, $\epsilon^2\varpi$) as ϖ itself. Thus, the formulæ for the cases $\theta = \epsilon^n\varpi$ can be obtained by simple substitution.

The definition of $\gamma_\Phi(X^*, Y)$ when $\theta \equiv \epsilon\varpi \pmod{(k^\times)^2}$ is an instance of this; namely, by Lemma 6.2,

$$\begin{aligned} -\gamma_{\mathrm{un}}(s)\gamma_{\mathrm{ram}}(s) &= (-1)^{r'}\operatorname{sgn}_\epsilon(s)\cdot\operatorname{sgn}_\varpi(-s)G_\varpi(\Phi') \\ &= \operatorname{sgn}_{\epsilon\varpi}(-s)\cdot\operatorname{sgn}_\varpi(\epsilon)^{r'}G_\varpi(\Phi') \\ &= \operatorname{sgn}_{\epsilon\varpi}(-s)G_{\epsilon\varpi}(\Phi'), \end{aligned}$$

where we have used that $\operatorname{sgn}_\epsilon(-1) = 1$ and $\operatorname{sgn}_\varpi(\epsilon) = -1$.

We next define a constant $c_0(X^*)$ for use in Theorem 9.7 and 10.10. Those theorems (and Proposition 11.2) will show that, as the notation suggests, it is the coefficient of the trivial orbit in the expansion of the germ of $\hat\mu^G_{X^*}$ in terms of Fourier transforms of nilpotent orbital integrals (see [25, Theorem 5.11]).

Definition 6.10.

$$c_0(X^*) = \begin{cases} -2q^{-1}, & X^* \text{ split} \\ -q^{-1}, & X^* \text{ unramified} \\ -\frac{1}{2}q^{-2}(q+1), & X^* \text{ ramified.} \end{cases}$$

Recall that $\hat\mu^G_{X^*}$ is defined in terms of the measure $\mathrm{d}\dot g$ of Proposition 11.2; and note that, in the notation of that proposition,

$$c_0(X^*) = (q-1)^{-1}\operatorname{meas}_{\mathrm{d}\dot g}(\dot K)$$

whenever X^* is elliptic.

7. Bessel functions

Our strategy for computing Fourier transforms of orbital integrals is to reduce them to p-adic Bessel functions (see Proposition 8.11, (9.3), and (10.2)). In this context, we are referring to the complex-valued Bessel functions defined in [42, §4], not the p-adic-valued ones defined in [20].

The definition of these functions depends on an additive character, denoted by χ in [42], and a multiplicative character, there denoted by π, of k. For internal consistency, we will instead denote the additive character by Φ and the multiplicative character by χ; but, for consistency with their work, we shall require throughout this section that $\mathrm{d}(\Phi) = -1$, i.e., that Φ is trivial on R but not on $\wp^{-1}$.

Definition 7.1 ([42, (4.1)])**.** For $\chi \in \widehat{k^\times}$, the *p-adic Bessel function of order* χ is given by

$$J_\chi(u, v) = \mathrm{P}\!\!\int_{k^\times}\Phi(ux + vx^{-1})\chi(x)\mathrm{d}^\times x \quad \text{for } u, v \in k^\times,$$

where $\mathrm{d}^\times x$ is the Haar measure on $k^\times$ fixed in Definition 2.1. We also put $J^\theta_\chi = \frac{1}{2}(J_\chi + J_{\chi\operatorname{sgn}_\theta})$, with notation as in Definition 3.2.

The locally constant K-Bessel function $K(z \mid \chi)$ of [54, Definition 3.2] is $J_\chi(\varpi^t, \varpi^t)$ (in the notation of that definition), where ϖ is a uniformiser.

Note that, for $\chi \neq 1$, it is natural to extend the Bessel function by putting $J_\chi(u,0) = \chi(u)^{-1}\Gamma(\chi)$ and $J_\chi(0,v) = \chi(v)\Gamma(\chi^{-1})$, where the Γ-factors are as in [42, §3], and that, under some conditions on χ, we can even define $J_\chi(0,0)$ (either as 0 or the sum of a geometric series); but we do not need to do this.

The notation J_χ^θ arises naturally in our computations; see Proposition 8.11.

Definition 7.2. We say that a character $\chi \in \widehat{k^\times}$ is *mildly ramified* if χ is trivial on $1 + \wp$, but non-trivial on $k^\times$.

Since our orbital-integral calculations require information about J_χ only for χ mildly ramified, and since more precise information is available in that case in general, it is there that we focus our attention.

Notation 7.3. We fix the following notation for the remainder of the section.

- $u, v \in k^\times$;
- $m = -\operatorname{ord}(uv)$; and
- $\chi \in \widehat{k^\times}$.

This is consistent with Notation 8.6. After Proposition 7.5, we will assume that χ is mildly ramified.

Of particular interest to us later will be the cases where χ is an unramified twist of one of the characters $\operatorname{sgn}_{\theta'}$ of Definition 3.2 (i.e., is of the form $\nu^\alpha \operatorname{sgn}_\theta$ for some $\alpha \in \mathbb{C}$). Note that $\operatorname{sgn}_\epsilon = \nu^{\pi i/\ln(q)}$.

Theorem 7.4 (Theorems 4.8 and 4.9 of [42])**.**

$$J_\chi(u,v) = \begin{cases} \chi(v)\Gamma(\chi^{-1}) + \chi(u)^{-1}\Gamma(\chi), & m \le 1 \\ \chi(u)^{-1}F_\chi(m/2, uv), & m \ge 2 \text{ and } m \text{ even} \\ 0, & m > 2 \text{ and } m \text{ odd,} \end{cases}$$

where the Γ*-factors are as in* [42, §3]*, and*

$$F_\chi(m/2, uv) := \int_{\operatorname{ord}(x) = -m/2} \Phi(x + uvx^{-1})\chi(x)\mathrm{d}^\times x.$$

The Γ-factor tables of [42, Theorem 3.1], together with Lemma 6.3, mean that we understand $J_\chi(u,v)$ completely when $m < 2$, but further calculation is necessary in the remaining cases.

Proposition 7.5. *If*

- $h \in \mathbb{Z}_{>0}$,
- χ *is trivial on* $1 + \wp^h$*, and*
- $m \ge 4h - 1$,

then $J_\chi(uv) = 0$ *if* $uv \notin (k^\times)^2$*; and, if* $w \in k^\times$ *satisfies* $uv = w^2$*, then*

$$J_\chi(u,v) = q^{-m/4}\chi(u^{-1}w) \times \begin{cases} \Phi(2w) + \chi(-1)\Phi(-2w), & 4 \mid m \\ \operatorname{sgn}_\varpi(w)G_\varpi(\Phi)\big(\Phi(2w) + (\chi\operatorname{sgn}_\varpi)(-1)\Phi(-2w)\big), & 4 \nmid m \end{cases}$$

Proof. If m is odd, then the vanishing result follows from Theorem 7.4, so we assume that m is even. In this case, $m \ge 4h$; and, by Theorem 7.4, $J_\chi(u,v) = \chi(u)^{-1}F_\chi(m/2, uv)$.

We evaluate the integral defining $F_\chi(m/2, uv)$ by splitting it into pieces. Write

$$S_{uv} = \bigl\{x \in k : \operatorname{ord}(x) = -m/2 \text{ and } \operatorname{ord}(x - uvx^{-1}) < -m/2 + h\bigr\}$$

and

$$T_{uv} = \bigl\{x \in k : \operatorname{ord}(x) = -m/2 \text{ and } \operatorname{ord}(x - uvx^{-1}) \geq -m/2 + h\bigr\}.$$

Note that both S_{uv} and T_{uv} are invariant under multiplication by $1 + \wp$; and that, if $x \in T_{uv}$, then $uv \in x^2(1 + \wp^h) \subseteq (k^\times)^2$. We claim that the relevant integral may be taken over only T_{uv}.

If $X \in \wp^{m/2-h}$, then we have by Lemma 2.6 and the fact that $2(m/2-h) \geq m/2$ that

$$\mathrm{c}(X) \equiv 1 + 2X \pmod{\wp^{m/2}} \quad \text{and} \quad \mathrm{c}(X)^{-1} \equiv 1 - 2X \pmod{\wp^{m/2}},$$

so

$$\begin{aligned}
&\int_{S_{uv}} \Phi(x + uvx^{-1})\chi(x)\mathrm{d}^\times x \\
&\qquad = (\star)\int_{\wp^{m/2-h}} \int_{S_{uv}} \Phi\bigl(x\cdot\mathrm{c}(X) + uvx^{-1}\cdot\mathrm{c}(X)^{-1}\bigr)\chi\bigl(x\cdot\mathrm{c}(X)\bigr)\mathrm{d}^\times x\,\mathrm{d}X \\
&\qquad = (\star)\int_{S_{uv}} \Phi(x + uvx^{-1})\chi(x)\int_{\wp^{m/2-h}} \Phi_{2(x-uvx^{-1})}(X)\mathrm{d}X\,\mathrm{d}^\times x,
\end{aligned}$$

where $(\star) = \operatorname{meas}_{\mathrm{d}X}(\wp^{m/2-h})^{-1}$ is a constant. We used that Φ is trivial on $x\wp^{m/2} \cup uvx^{-1}\wp^{m/2} \subseteq R$ and χ is trivial on $\mathrm{c}(\wp^{m/2-h}) = 1 + \wp^{m/2-h} \subseteq 1 + \wp^h$. By (2.3), we have that $\mathrm{d}(\Phi_{2(x-uvx^{-1})}) > m/2 - h + 1$ (i.e., $\Phi_{2(x-uvx^{-1})}$ is a non-trivial character on $\wp^{m/2-h}$) whenever $x \in S_{uv}$, so the inner integral is 0. This shows that, as desired, the integral defining $F_\chi(m/2, uv)$ may be taken over only T_{uv}.

If $uv \notin (k^\times)^2$, then $T_{uv} = \emptyset$, so $J_\chi(u,v) = \chi(u)^{-1}F_\chi(m/2, uv) = 0$; whereas, if $w \in k^\times$ satisfies $w^2 = uv$, then $T_{uv} = w(1 + \wp^h) \sqcup -w(1 + \wp^h)$, so

$$(*)\quad J_\chi(u,v) = \chi(u)^{-1}\Bigl(\int_{w(1+\wp^h)} \Phi(x+uvx^{-1})\chi(x)\mathrm{d}^\times x + \int_{-w(1+\wp^h)} \Phi(x+uvx^{-1})\chi(x)\mathrm{d}^\times x\Bigr).$$

Note that $\operatorname{ord}(w) = -m/2$.

We show a detailed calculation of the first integral; of course, that of the second is identical. Note that the integral no longer involves χ. By Lemma 2.6 again, we have that $X \mapsto w\cdot\mathrm{c}(X)$ is a measure-preserving bijection from $\wp^h$ to $w(1 + \wp^h)$, so

$$\int_{w(1+\wp^h)} \Phi(x + uvx^{-1})\chi(x)\mathrm{d}^\times x = \chi(w)\int_{\wp^h} \Phi_w\bigl(\mathrm{c}(X) + \mathrm{c}(X)^{-1}\bigr)\mathrm{d}X,$$

where we have used that $uvw^{-1} = w$ and again that χ is trivial on $\mathrm{c}(\wp^h) = 1 + \wp^h$. We will evaluate the latter integral by breaking it into 'shells' on which $\operatorname{ord}(X)$ is constant, using the following facts. Note that, by direct computation (and Definition 2.5),

$$\mathrm{c}(X) + \mathrm{c}(X)^{-1} = 2\mathrm{c}(X^2)$$

for $X \in k \setminus \{1\}$. If $\operatorname{ord}(X) = i$ and $\operatorname{ord}(Y) = j$, then we have by Lemma 2.6 once more that

$$\mathrm{c}\bigl((X + Y)^2\bigr) \equiv \mathrm{c}(X^2 + 2XY) \pmod{\wp^{2j}}$$

and

$$\mathrm{c}(X^2 + 2XY) \equiv \mathrm{c}(X^2) + 4XY \pmod{\wp^{2j}}.$$

(In fact, the second congruence could be made much finer, but that would be of no use here.)

In particular, fix $i \geq h$ with $2i < m/2-1$, so that $\mathrm{d}(\Phi) = m/2-1 < 2(m/2-1-i)$ (i.e., Φ is trivial on $\wp^{2(m/2-1-i)}$). Then

$$\begin{aligned}
&\int_{\mathrm{ord}(X)=i} \Phi_w\bigl(\mathrm{c}(X) + \mathrm{c}(X)^{-1}\bigr)\mathrm{d}X \\
&\qquad = (\star) \int_{\wp^{m/2-1-i}} \int_{\mathrm{ord}(X)=i} (\Phi_{2w} \circ \mathrm{c})\bigl((X+Y)^2\bigr)\mathrm{d}X\,\mathrm{d}Y \\
&\qquad = (\star) \int_{\mathrm{ord}(X)=i} (\Phi_{2w} \circ \mathrm{c})(X^2) \int_{\wp^{m/2-1-i}} \Phi_{8wX}(Y)\mathrm{d}Y\,\mathrm{d}X,
\end{aligned}$$

where $(\star) = \mathrm{meas}(\wp^{m/2-1-i})^{-1}$ is a constant. Since $\mathrm{d}(\Phi_{8wX}) = \mathrm{d}(\Phi_w) - \mathrm{ord}(8X) = m/2-1-i$, the inner integral is 0.

Note that $\lceil (m/2-1)/2 \rceil \geq h$. We have thus shown that

$$J_\chi(u,v) = \int_{\wp^{\lceil (m/2-1)/2 \rceil}} (\Phi_{2w} \circ \mathrm{c})(X^2)\mathrm{d}X.$$

If $m/2$ is even, then the integral is over $\wp^{m/4}$, and $\mathrm{c}(X^2) \equiv 1 \pmod{\wp^{m/2} \subseteq \ker \Phi_{2w}}$ for all $X \in \wp^{m/4}$. Thus, in that case,

$$J_\chi(u,v) = \mathrm{meas}_{\mathrm{d}X}(\wp^{m/4})\Phi_{2w}(1) = q^{-m/4}\Phi(2w).$$

If $m/2$ is odd, then the integral is over $\wp^{m/4-1/2}$, and $\mathrm{c}(X^2) \equiv 1 + 2X^2 \pmod{\wp^{m/2}}$ for all $X \in \wp^{m/4-1/2}$. Thus, in that case,

$$\begin{aligned}
J_\chi(u,v) &= \mathrm{meas}_{\mathrm{d}X}(\wp^{m/4+1/2})\Phi_{2w}(1) \sum_{X \in \wp^{m/4-1/2}/\wp^{m/4+1/2}} \Phi_{4w}(X^2) \\
&= q^{-m/4}\Phi(2w)q^{-1/2} \sum_{X \in R/\wp} \Phi_{4w\varpi^{m/2-1}}(X^2).
\end{aligned}$$

By Lemma 6.2, and the fact that $m/2$ is odd, this can be re-written as

$$q^{-m/4}\Phi(2w)\,\mathrm{sgn}_\varpi(-1)^{m/2-1}G_\varpi(\Phi_{4w}) = q^{-m/4}\Phi(2w)\,\mathrm{sgn}_\varpi(w)G_\varpi(\Phi).$$

The result now follows from $(*)$. $\square$

From now on, we assume that χ is mildly ramified. In particular, we may take $h = 1$, so that Proposition 7.5 holds whenever $m > 2$.

Definition 7.6. For

- $\xi \in \mathfrak{f}^\times$,
- $\overline{\Phi}$ an (additive) character of $\mathfrak{f}$, and
- $\overline{\chi}$ a (multiplicative) character of $\mathfrak{f}^\times$,

we define the corresponding *twisted Kloosterman sum* by

$$K(\overline{\chi}, \overline{\Phi}; \xi) := \sum_{x \in \mathfrak{f}^\times} \overline{\Phi}(x + \xi x^{-1})\overline{\chi}(x).$$

Proposition 7.7. *If $m = 2$, then*

$$J_\chi(u, v) = q^{-1}\chi(u\varpi)^{-1}K(\overline{\chi}, \overline{\Phi}; \xi),$$

Here,

- ξ *is the image in* $\mathfrak{f}^\times$ *of* $\varpi^2 uv \in R^\times$,
- $\overline{\Phi}$ *is the (additive) character of* $\mathfrak{f} = R/\wp$ *arising from the restriction to* R *of the depth-*0 *character* $\Phi_{\varpi^{-1}}$ *of* k, *and*
- $\overline{\chi}$ *is the (multiplicative) character of* $\mathfrak{f}^\times \cong R^\times/1 + \wp$ *arising from the restriction to* $R^\times$ *of* χ.

Proof. By Theorem 7.4,

$$\begin{aligned}
\chi(u\varpi)J_\chi(u, v) &= \chi(\varpi)\int_{\operatorname{ord}(x)=-1} \Phi(x + uvx^{-1})\chi(x)\mathrm{d}^\times x \\
&= \int_{R^\times} \Phi(\varpi^{-1}x + uv\cdot\varpi x^{-1})\chi(x)\mathrm{d}^\times x \\
&= \operatorname{meas}_{\mathrm{d}^\times x}(1 + \wp)\sum_{x\in R^\times/1+\wp} \Phi_{\varpi^{-1}}(x + \varpi^2 uvx^{-1})\chi(x)\mathrm{d}^\times x.
\end{aligned}$$

Since $\operatorname{meas}_{\mathrm{d}^\times x}(1 + \wp) = q^{-1}$, the result follows. □

Corollary 7.8. *Suppose that $m = 2$. Then*

$$\begin{aligned}
J_{\nu^\alpha}(u, v) &= q^{\alpha-1}\left|u\right|^{-\alpha}\sum_{\substack{c\in\wp^{-1}/R\\ c^2\ne uv}} \Phi(2c)\operatorname{sgn}_\varpi(c^2 - uv) \\
J_{\nu^\alpha \operatorname{sgn}_\varpi}(u, v) &= q^{\alpha-1/2}\left|u\right|^{-\alpha}\operatorname{sgn}_\varpi(v)G_\varpi(\Phi)\sum_{\substack{c\in\wp^{-1}/R\\ c^2=uv}} \Phi(2c)
\end{aligned}$$

for $\alpha \in \mathbb{C}$.

Proof. If $\chi = \nu^\alpha$, then $\overline{\chi} = 1$, so [32, Theorem 5.47] gives that

$$\begin{aligned}
K(\overline{\chi}, \overline{\Phi}; \xi) &= \sum_{\substack{c\in\mathfrak{f}\\ c^2\ne\xi}} \overline{\Phi}(2c)\operatorname{sgn}_{\mathfrak{f}}(c^2 - \alpha) \\
&= \sum_{\substack{c\in R/\wp\\ c^2\ne\varpi^2 uv}} \overline{\Phi}(2c)\operatorname{sgn}_\varpi(c^2 - \varpi^2 uv) \\
&= \sum_{\substack{c\in\wp^{-1}/R\\ c^2\ne uv}} \Phi(2c)\operatorname{sgn}_\varpi(c^2 - uv).
\end{aligned}$$

(Note that our $\overline{\Phi}$ is their χ, and that they write $K(\chi; a, b)$ where we write $K(\overline{\Phi}, 1; ab)$.)

If $\chi = \nu^\alpha \operatorname{sgn}_\varpi$, then $\overline{\chi} = \operatorname{sgn}_{\mathfrak{f}}$, so [32, Exercises 5.84–85] gives that

$$\begin{aligned}
K(\overline{\chi}, \overline{\Phi}; \xi) &= \operatorname{sgn}_{\mathfrak{f}}(\xi)G(\operatorname{sgn}_{\mathfrak{f}}, \overline{\Phi})\sum_{\substack{c\in\mathfrak{f}\\ c^2=\xi}} \overline{\Phi}(2c) \\
&= \operatorname{sgn}_\varpi(uv)G(\operatorname{sgn}_{\mathfrak{f}}, \overline{\Phi})\sum_{\substack{c\in\wp^{-1}/R\\ c^2=uv}} \Phi(2c),
\end{aligned}$$

where $G(\operatorname{sgn}_{\mathfrak f}, \overline{\Phi}) = \sum_{X \in \mathfrak f^\times} \overline{\Phi}(X) \operatorname{sgn}_{\mathfrak f}(X)$. (Note that our $\overline{\Phi}$ is their χ and our $\overline{\chi}$ their η, and that they write $K(\eta, \chi; 1, \xi)$ where we write $K(\overline{\chi}, \overline{\Phi}; \xi)$.) Since $\mathrm{d}(\Phi) = -1$, Lemma 6.2 gives that $G(\operatorname{sgn}_{\mathfrak f}, \overline{\Phi}) = q^{1/2} \operatorname{sgn}_\varpi(-1) G_\varpi(\Phi)$.

The result now follows from Proposition 7.7. □

We now state an apparently rather specialised corollary, which nonetheless turns out to be sufficient to simplify many of our 'shallow' computations (see §9.1 and §10.1).

Corollary 7.9. *If $m \ge 2$ and $\operatorname{ord}(u) = \operatorname{ord}(v)$, then $J_{\nu^\alpha\chi}(u, v)$ is independent of $\alpha \in \mathbb{C}$; in particular,*

$$J^\epsilon_\chi(u, v) = J_\chi(u, v) \quad \text{and} \quad J^\varpi_\chi(u, v) = J^\varpi_{\chi \operatorname{sgn}_\epsilon}(u, v).$$

If $m \ge 2$ and $\operatorname{ord}(u) = \operatorname{ord}(v) + 2$, then $J_{\nu^\alpha\chi}(u, v) = q^\alpha J_\chi(u, v)$; in particular,

$$J^\epsilon_\chi(u, v) = 0 \quad \text{and} \quad J^\varpi_\chi(u, v) = -J^\varpi_{\chi \operatorname{sgn}_\epsilon}(u, v).$$

Proof. Suppose that $m > 2$. If $uv \notin (k^\times)^2$, then $J_{\nu^\alpha\chi}(u, v) = 0$ for all $\alpha \in \mathbb{C}$. If $uv = w^2$, then the only dependence on α in Proposition 7.5 is via the factor $\chi(u^{-1}w)$. If $\operatorname{ord}(u) = \operatorname{ord}(v)$, then also $\operatorname{ord}(w) = \operatorname{ord}(u)$, so $\nu^\alpha(u^{-1}w) = 1$. If $\operatorname{ord}(u) = \operatorname{ord}(v) + 2$, then $\operatorname{ord}(w) = \operatorname{ord}(u) - 1$, so $\nu^\alpha(u^{-1}w) = q^\alpha$.

Now suppose that $m = 2$, i.e., that $\operatorname{ord}(uv) = -2$. Since $\overline{\nu^\alpha\chi} = \overline{\chi}$, the only dependence on α in Proposition 7.7 is via the factor $\chi(u\varpi)^{-1}$. If $\operatorname{ord}(u) = \operatorname{ord}(v)$, then $\operatorname{ord}(u) = -1$, so $\nu^\alpha(u\varpi) = 1$. If $\operatorname{ord}(u) = \operatorname{ord}(v) + 2$, then $\operatorname{ord}(u) = 0$, so $\nu^\alpha(u\varpi) = q^{-\alpha}$. □

8. A mock-Fourier transform

We begin by introducing a function $M^G_{X^*}$ specified by an integral formula (see Definition 8.4) reminiscent of the usual one for (the function representing) $\hat\mu^G_{X^*}$ (see [3, Theorem A.1.2]). We will eventually show (see Proposition 11.2) that it is actually *equal* to $\hat\mu^G_{X^*}$, but first we spend some time computing it.

In the notation of Definition 4.1, we have

$$\operatorname{tr} g \cdot \sqrt{\theta} \cdot g^{-1} \cdot \sqrt{\theta'} = N_\theta(\alpha) \cdot \theta' + N_\theta(\gamma), \tag{8.1}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\alpha = a + b\sqrt{\theta}$, and $\gamma = c + d\sqrt{\theta}$. Since $1 = ad - bc = \operatorname{Im}_\theta(\overline{\alpha} \cdot \gamma)$, we have that $\gamma = \overline{\alpha}^{-1} \cdot (t + \sqrt{\theta})$ for some $t \in k$; specifically, $t = \operatorname{Re}_\theta(\overline{\alpha} \cdot \gamma) = ac - bd\theta$. This calculation motivates the definition of the following map.

Definition 8.2. We define $\varphi_\theta : G \to k_\theta^\times \times k$ by

$$\varphi_\theta(g) = (a + b\sqrt{\theta}, ac - bd\theta)$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$.

Note that φ_θ is a bi-analytic map (of k-manifolds), with inverse

$$(\alpha, t) \mapsto \begin{pmatrix} \operatorname{Re}_\theta(\alpha) & \operatorname{Im}_\theta(\alpha) \\ N_\theta(\alpha)^{-1}\bigl(t \cdot \operatorname{Re}_\theta(\alpha) + \theta \cdot \operatorname{Im}_\theta(\alpha)\bigr) & N_\theta(\alpha)^{-1}\bigl(\operatorname{Re}_\theta(\alpha) + t \cdot \operatorname{Im}_\theta(\alpha)\bigr) \end{pmatrix}.$$

It is *not* an isomorphism, but its restrictions to T_θ, A, and $\bigl\{\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right) : b \in k\bigr\}$ are isomorphisms onto $C_\theta \times \{0\}$, $k^\times \times \{0\}$, and $\{1\} \times k$, respectively. In fact, the next lemma says a bit more.

Lemma 8.3. *If $g \in G$ satisfies $\varphi(g) = (\alpha, t)$, and*

- *$h \in T_\theta$ is identified with $\eta \in C_\theta$,*
- *$a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ (with $\lambda \in k^\times$), and*
- *$\overline{u} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ (with $b \in k$),*

then

$$\varphi_\theta(gh) = (\alpha\eta, t),$$
$$\varphi_\theta(ag) = (\lambda\alpha, t),$$

and

$$\varphi_\theta(\overline{u}g) = (\alpha, t + N_\theta(\alpha)b).$$

Proof. This is a straightforward computation. □

Now we are in a position to define our 'mock orbital integral'. Again, Proposition 11.2 will eventually show that it is actually equal to the function in which we are interested.

Definition 8.4. For $\alpha \in k_\theta^\times$ and $t \in k$, put

$$\langle X^*, Y\rangle_{\alpha,t} := \beta s\bigl(N_\theta(\alpha) \cdot \theta' + N_\theta(\alpha)^{-1} \cdot \theta - N_\theta(\alpha)^{-1} \cdot t^2\bigr).$$

Notice that the dependence on α is only via $N_\theta(\alpha)$. Thus, we may define

$$M_{X^*}^G(Y) := ⨍_{k_\theta^\times / C_\theta} ⨍_k \Phi(\langle X^*, Y\rangle_{\alpha,t}) \mathrm{d}t \, \mathrm{d}^\times \dot{\alpha},$$

where

$$⨍_k f(x)\mathrm{d}t := \sum_{n\in\mathbb{Z}} \int_{\operatorname{ord}(x)=n} f(x)\mathrm{d}t$$
$$⨍_{k^\times} f(x)\mathrm{d}^\times x := \sum_{n\in\mathbb{Z}} \int_{\operatorname{ord}(x)=n} f(x)\mathrm{d}^\times x$$

and

$$⨍_{k_\theta^\times / C_\theta} (f \circ N_\theta)(\alpha)\mathrm{d}^\times \dot{\alpha} := ⨍_{k^\times} [N_\theta(k^\times)](x) f(x) \mathrm{d}^\times x$$

(for those $f \in C^\infty(k)$ for which the sum converges) are 'principal-value' integrals, as in [42, p. 282]. Here, $\mathrm{d}t$ and $\mathrm{d}^\times x$ are the measures of Definition 2.1, and $[S]$ denotes the characteristic function of S.

By (8.1) (and Notations 5.1 and 5.8), we have that

$$\langle X^*, Y\rangle_{\alpha,t} = \langle \mathrm{Ad}^*(g)X^*, Y\rangle \quad \text{when } \varphi_\theta(g) = (\alpha, t), \tag{8.5}$$

where the pairing $\langle\cdot,\cdot\rangle$ on the right is the usual pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Notation 8.6. $u = \varpi^{-(r'+1)} s\theta'$, $v = \varpi^{-(r'+1)} s\theta$, and $m = -\operatorname{ord}(uv)$.

This is a special case of Notation 7.3. These particular values of u and v will be fixed for the remainder of the paper. It follows that

$$uv = (\varpi^{-(r'+1)} s)^2 \cdot \theta\theta', \tag{8.7}$$

so

$$uv \in (k^\times)^2 \Leftrightarrow \theta\theta' \in (k^\times)^2; \tag{8.8}$$

and we use Lemma 5.9 to compute

$$\operatorname{ord}(u) = -(r'+1) + \operatorname{ord}(s\theta') = -\bigl(r'+1+\tfrac{1}{2}\operatorname{ord}(\theta')\bigr) + \mathrm{d}(Y) \tag{8.9}$$

and

$$m = 2(r'+1) - \operatorname{ord}(s^2\theta') - \operatorname{ord}(\theta) = 2(r'+1-\mathrm{d}(Y)) - \operatorname{ord}(\theta). \tag{8.10}$$

8.1. **Mock-Fourier transforms and Bessel functions.** We can now evaluate the integral occurring in Definition 8.4 in terms of Bessel functions—or, rather, the sums J^θ_χ of Definition 7.1.

Proposition 8.11.

$$\begin{aligned} M^G_{X^*}(Y) = \tfrac{1}{2}\bigl|s\bigr|^{-1/2} q^{-(r'+1)/2} \times \\ \Bigl(\bigl(J^\theta_{\nu^{1/2}}(u,v) + \gamma_{\mathrm{un}}(s) J^\theta_{\nu^{1/2}\operatorname{sgn}_\epsilon}(u,v)\bigr) + \\ \gamma_{\mathrm{ram}}(s)\bigl(J^\theta_{\nu^{1/2}\operatorname{sgn}_\varpi}(u,v) - \gamma_{\mathrm{un}}(s) J^\theta_{\nu^{1/2}\operatorname{sgn}_{\epsilon\varpi}}(u,v)\bigr)\Bigr), \end{aligned}$$

where J^θ_χ is as in Definition 7.1, and $\gamma_{\mathrm{un}}(s)$ and $\gamma_{\mathrm{ram}}(s)$ are as in Definition 6.5.

Proof. Recall the notation $\Phi' = \Phi_\beta$ from Notation 5.2. By Definition 8.4,

$$\begin{aligned} & M^G_{X^*}(Y) \\ & = \fint_{k^\times_\theta/C_\theta} \Phi'_s\bigl(N_\theta(\alpha)\cdot\theta' + N_\theta(\alpha)^{-1}\cdot\theta\bigr)\cdot \fint_k \Phi'(-sN_\theta(\alpha)^{-1}t^2)\mathrm{d}t\,\mathrm{d}^\times\dot\alpha \\ & = q^{-(r'+1)/2}\fint_{k^\times}[N_\theta(k^\times_\theta)](x)j(\theta',\theta;x)\mathcal{H}(\Phi',-sx^{-1})\mathrm{d}^\times x, \end{aligned} \tag{$*$}$$

where

- $j(\theta',\theta;x) := \Phi'_s(\theta' x + \theta x^{-1}) = \Phi\bigl(\beta s(\theta' x + \theta x^{-1})\bigr)$ for $x \in k^\times$; and
- $\mathcal{H}(\Phi', b) = \fint_k \Phi'(bt^2)\mathrm{d}_{\Phi'}t$ for $b \in k^\times$ is as in [48, p. 6].

In particular, $\mathrm{d}_{\Phi'}t$ is the Φ'-self-dual Haar measure on k; by [48, p. 5], it satisfies $\mathrm{d}t = q^{-(r'+1)/2}\mathrm{d}_{\Phi'}t$. This is the reason for the appearance of $q^{-(r'+1)/2}$ on the last line of the computation.

The significance of j is that integrating it against a (multiplicative) character χ of $k^\times$ corresponds to evaluating a Bessel function of order χ, in the sense of Definition 7.1. To be precise, note that our character Φ' has depth r', not -1, so that we must work instead with $\Phi'_{\varpi^{r'+1}}$. Then

$$j(\theta',\theta;x) = \Phi'_{\varpi^{r'+1}}\bigl((\varpi^{-(r'+1)}s\theta')x + (\varpi^{-(r'+1)}s\theta)x^{-1}\bigr) = \Phi'_{\varpi^{r'+1}}(ux + vx^{-1}),$$

where (u,v) is as in Notation 8.6, so

$$\fint_{k^\times} j(\theta',\theta;x)\chi(x)\mathrm{d}^\times x = J_\chi(u,v) \tag{$\dagger$}$$

for $\chi \in \widehat{k^\times}$.

Now note that $\frac{1}{2}(1+\operatorname{sgn}_\theta)$ is the characteristic function of $N_\theta(k^\times_\theta)$, so we may re-write $(*)$ as

$$q^{-(r'+1)/2}\fint_{k^\times}\tfrac{1}{2}(1+\operatorname{sgn}_\theta(x))\cdot j(\theta',\theta;x)\mathcal{H}(\Phi',-sx^{-1})\mathrm{d}^\times x. \tag{$**$}$$

By [48, Lemma 1.3.2] and Lemma 6.2, we have

$$\mathcal{H}(\Phi', b) = \left|b\right|^{-1/2} \begin{cases} \operatorname{sgn}_\varpi(b) G_\varpi(\Phi'), & r' - \operatorname{ord}(b) \text{ even} \\ 1, & r' - \operatorname{ord}(b) \text{ odd}. \end{cases}$$

We find it useful to offer a description of $\mathcal{H}(\Phi', b)$ without explicit use of cases. As above, we note that $\frac{1}{2}(1+(-1)^n \operatorname{sgn}_\epsilon)$ is the characteristic function of $\bigl\{b \in k^\times : \operatorname{ord}(b) \equiv n \pmod 2\bigr\}$, so that we may re-write

$$\mathcal{H}(\Phi', b) = \tfrac{1}{2}(1 + (-1)^{r'} \operatorname{sgn}_\epsilon(b)) \operatorname{sgn}_\varpi(b) G_\varpi(\Phi') + \tfrac{1}{2}(1 - (-1)^{r'} \operatorname{sgn}_\epsilon(b)).$$

Plugging this into $(**)$, with $b = -st^{-1}$, gives

$$\begin{aligned} M^G_{X^*}(Y) = \tfrac{1}{2}\left|s\right|^{-1/2} q^{-(r'+1)/2} \fint_{k^\times} & \tfrac{1}{2}(1 + \operatorname{sgn}_\theta(x)) \times \\ & \bigl((1 - \gamma_{\mathrm{un}}(s) \operatorname{sgn}_\epsilon(x))\gamma_{\mathrm{ram}}(s) \operatorname{sgn}_\varpi(x) + \\ & \qquad (1 + \gamma_{\mathrm{un}}(s) \operatorname{sgn}_\epsilon(x))\bigr) \times \\ & \left|x\right|^{1/2} j(\theta', \theta; x) \mathrm{d}^\times x. \end{aligned}$$

Expanding the product and applying (†) gives the desired formula. □

8.2. **'Deep' Bessel functions.** By Proposition 8.11, one approach to computing $M^G_{X^*}(Y)$ (hence $\hat{\mu}^G_{X^*}(Y)$, by Proposition 11.2) is to evaluate many Bessel functions, and this is exactly what we do. As Theorem 7.4 makes clear, the behaviour of Bessel functions is more predictable when $m < 2$ than otherwise. We introduce a convenient, but temporary, shorthand for referring to Bessel functions in this range; we will only use it in this section, and §§9.2 and 10.3.

Notation 8.12. We define

$$[A; B]_{\theta, r'}(\theta') := \left|\theta\right|^{1/2} A + q^{-(r'+1)} \left|D_{\mathfrak{g}}(Y)\right|^{-1/2} B(\theta').$$

We will usually suppress the subscript on $[A; B]$, and will sometimes write

$$[A; B(1), B(\epsilon), B(\varpi), B(\epsilon\varpi)](\theta')$$

for the same quantity.

Proposition 8.13. *With the notation of Notations 5.2, 5.8, and 8.6, and Definition 6.5, if $m < 2$, then*

$$\begin{aligned} & \left|s\right|^{-1/2} q^{-(r'+1)/2} J_{\nu^{1/2}\chi}(u, v) \\ & \qquad = \begin{cases} \bigl[Q_3(q^{-1/2}); 1\bigr](\theta'), & \chi = 1 \\ \gamma_{\mathrm{un}}(s) \bigl[\operatorname{sgn}_\epsilon(\theta) Q_3(-q^{-1/2}); \operatorname{sgn}_\epsilon\bigr](\theta'), & \chi = \operatorname{sgn}_\epsilon \\ \gamma_{\mathrm{ram}}(s)^{-1} \bigl[\operatorname{sgn}_\varpi(\theta) q^{-1}; \operatorname{sgn}_\varpi\bigr](\theta'), & \chi = \operatorname{sgn}_\varpi \\ -\gamma_{\mathrm{un}}(s)\gamma_{\mathrm{ram}}(s)^{-1} \bigl[\operatorname{sgn}_{\epsilon\varpi}(\theta) q^{-1}; \operatorname{sgn}_{\epsilon\varpi}\bigr](\theta'), & \chi = \operatorname{sgn}_{\epsilon\varpi}, \end{cases} \end{aligned}$$

where

$$Q_3(T) = -T(T^2 + T + 1).$$

The unexpected factor $\left|s\right|^{-1/2} q^{-(r'+1)/2}$ above crops up repeatedly in calculations (see, for example, Proposition 8.11), so it simplifies matters to include it in this calculation.

Proof. By Theorem 7.4 and Lemma 5.9,

$$\begin{aligned}
J_{\nu^{1/2}\chi}(u,v) &= (\nu^{1/2}\chi)(v)\Gamma(\nu^{-1/2}\chi) + (\nu^{-1/2}\chi)(u)\Gamma(\nu^{1/2}\chi) \\
&= (\nu^{1/2}\chi)(v\theta^{-1}) \times \\
&\qquad \bigl((\nu^{1/2}\chi)(\theta)\Gamma(\nu^{-1/2}\chi) + (\nu^{-1/2}\chi)(uv\theta^{-1})\Gamma(\nu^{1/2}\chi)\bigr) \\
&= \bigl|s\bigr|^{1/2} q^{(r'+1)/2}\chi(\varpi^{r'+1}s)\bigl[\chi(\theta)\Gamma(\nu^{-1/2}\chi); \Gamma(\nu^{1/2}\chi)\cdot\chi\bigr](\theta')
\end{aligned}$$

whenever $\chi^2 = 1$.

In particular, upon using [42, Theorem 3.1(i, ii)] to compute the Γ-factors, we see that $\bigl|s\bigr|^{-1/2}q^{-(r'+1)/2}J_{\nu^{1/2}\chi}(u,v)$ is given by

$$(*)\qquad \begin{cases}
\bigl[Q_3(q^{-1/2}); 1\bigr](\theta'), & \chi = 1 \\
\gamma_{\mathrm{un}}(s)\bigl[\operatorname{sgn}_\epsilon(\theta)Q_3(-q^{-1/2}); \operatorname{sgn}_\epsilon\bigr](\theta'), & \chi = \operatorname{sgn}_\epsilon \\
\operatorname{sgn}_\varpi(\varpi^{r'+1}s)\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)\bigl[\operatorname{sgn}_\varpi(\theta)q^{-1}; \operatorname{sgn}_\varpi\bigr](\theta'), & \chi = \operatorname{sgn}_\varpi \\
\gamma_{\mathrm{un}}(s)\operatorname{sgn}_\varpi(\varpi^{r'+1}s)\Gamma(\nu^{1/2}\operatorname{sgn}_{\epsilon\varpi})\bigl[\operatorname{sgn}_{\epsilon\varpi}(\theta)q^{-1}; \operatorname{sgn}_{\epsilon\varpi}\bigr](\theta'), & \chi = \operatorname{sgn}_{\epsilon\varpi}.
\end{cases}$$

By Theorem 3.1(ii) *loc. cit.* again, and the fact that $\operatorname{sgn}_{\epsilon\varpi} = \nu^{i\pi/\ln(q)}\operatorname{sgn}_\varpi$, we have that $\Gamma(\nu^{1/2}\operatorname{sgn}_{\epsilon\varpi}) = -\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi)$; and, by Lemma 6.3, Definition 6.5, and (6.7),

$$\begin{aligned}
&\operatorname{sgn}_\varpi(\varpi^{r'+1}s)\Gamma(\nu^{1/2}\operatorname{sgn}_\varpi) \\
&\qquad = \operatorname{sgn}_\varpi(-1)^{r'+1}\operatorname{sgn}_\varpi(s)\cdot\operatorname{sgn}_\varpi(-1)^{r'+1}G_\varpi(\Phi') \\
&\qquad = \operatorname{sgn}_\varpi(s)G_\varpi(\Phi') \\
&\qquad = \gamma_{\mathrm{ram}}(s)^{-1}.
\end{aligned}$$

This shows that $(*)$ reduces to the table in the statement. □

9. Split and unramified orbital integrals

Throughout this section, we have

$$\theta = 1 \text{ or } \theta = \epsilon, \quad \text{so that} \quad r' = r. \tag{9.1}$$

In the split case, $J_\chi^1 = J_\chi$ for $\chi \in \widehat{k^\times}$, so Proposition 8.11 gives

$$\begin{aligned}
M_{X^*}^G(Y) = \tfrac12\bigl|s\bigr|^{-1/2}&q^{-(r+1)/2} \times \\
&\Bigl(\bigl(J_{\nu^{1/2}}(u,v) + \gamma_{\mathrm{un}}(s)J_{\nu^{1/2}\operatorname{sgn}_\epsilon}(u,v)\bigr) + \\
&\quad \gamma_{\mathrm{ram}}(s)\bigl(J_{\nu^{1/2}\operatorname{sgn}_\varpi}(u,v) - \gamma_{\mathrm{un}}(s)J_{\nu^{1/2}\operatorname{sgn}_{\epsilon\varpi}}(u,v)\bigr)\Bigr),
\end{aligned}\tag{9.2}$$

In the unramified case, $J_\chi^\epsilon = J_{\chi\operatorname{sgn}_\epsilon}^\epsilon$ for $\chi \in \widehat{k^\times}$, so Proposition 8.11 gives

$$\begin{aligned}
M_{X^*}^G(Y) = \tfrac12\bigl|s\bigr|^{-1/2}&q^{-(r+1)/2} \times \\
&\bigl((1+\gamma_{\mathrm{un}}(s))J_{\nu^{1/2}}^\epsilon(u,v) + \gamma_{\mathrm{ram}}(s)(1-\gamma_{\mathrm{un}}(s))J_{\nu^{1/2}\operatorname{sgn}_\varpi}^\epsilon(u,v)\bigr).
\end{aligned}\tag{9.3}$$

By (6.8) and (6.7),

$$\operatorname{sgn}_\varpi(v)G_\varpi(\Phi'_{\varpi^{r+1}}) = \begin{cases}
\operatorname{sgn}_\varpi(-1)\gamma_{\mathrm{ram}}(s) = \gamma_{\mathrm{ram}}(s)^{-1}, & \theta = 1 \\
\operatorname{sgn}_\varpi(-\epsilon)\gamma_{\mathrm{ram}}(s) = -\gamma_{\mathrm{ram}}(s)^{-1}, & \theta = \epsilon.
\end{cases}\tag{9.4}$$

9.1. **Far from zero.** The results of this section are special cases for split and unramified orbital integrals of results of Waldspurger [55, Proposition VIII.1]. We shall prove analogues of these results for ramified orbital integrals in §10.1.

The qualitative behaviour of unramified orbital integrals does not change as we pass from elements of depth less than r to those of depth exactly r; this is unlike the situation for ramified orbital integrals. See §10.2.

Theorem 9.5. *If* $\mathrm{d}(X^*)+\mathrm{d}(Y) \le 0$ *and* X^* *is split or unramified, then* $M^G_{X^*}(Y) = 0$ *unless* X^* *and* Y *lie in* G*-conjugate tori.*

Proof. Recall that $\theta = 1$ if X^* is split, and $\theta = \epsilon$ if X^* is unramified.

By (8.10), $m \ge 2$; in fact, $m > 2$ (indeed, m is odd) unless $\operatorname{ord}(\theta')$ is even.

If $m > 2$, then Proposition 7.5 and (8.8) show that $M^G_{X^*}(Y) = 0$ unless $\theta\theta' \in (k^\times)^2$. By §4, it therefore suffices to consider the cases when $\theta = \epsilon$ and $\theta' = \varpi^2\epsilon$, i.e., X^* and Y lie in stably, but not rationally, conjugate tori; and when $m = 2$ and $\{\theta, \theta'\} = \{1, \epsilon\}$, i.e., one of X^* or Y is split, and the other unramified.

Suppose first that $\theta = \epsilon$ and $\theta' = \varpi^2\epsilon$, so that $\operatorname{ord}(u) = \operatorname{ord}(v) + 2$. By Corollary 7.9, (9.3) becomes $M^G_{X^*}(Y) = 0$.

Now suppose that $\{\theta, \theta'\} = \{1, \epsilon\}$ and $m = 2$. By Corollary 7.9, since $\operatorname{ord}(u) = \operatorname{ord}(v)$,

$$J_{\nu^{1/2}}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u, v) \quad \text{and} \quad J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_{\epsilon\varpi}}(u, v),$$

so (9.2) agrees with (9.3). We shall work with (9.3), since it is simpler.

By Corollary 7.8 and (8.8), $J_{\nu^\alpha\,\mathrm{sgn}_\varpi}(u, v) = 0$ for all $\alpha \in \mathbb{C}$; in particular, for $\alpha = 1/2$ and $\alpha = 1/2 + i\pi/\ln(q)$. By (8.10), $\operatorname{ord}(s) = r$, so, by Definition 6.5, $\gamma_{\mathrm{un}}(s) = -1$, and (9.3) (hence also (9.2)) becomes

$$M^G_{X^*}(Y) = \tfrac{1}{2}\left|s\right|^{-1/2} J^\epsilon_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u, v) = 0. \qquad \square$$

Theorem 9.6. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) \le 0$ *and* X^* *and* Y *lie in a common split or unramified torus* $\mathbf{T}$ *(with* $T = \mathbf{T}(k)$*), then*

$$M^G_{X^*}(Y) = q^{-(r+1)}\left|D_{\mathfrak{g}}(Y)\right|^{-1/2}\gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G,T)} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle),$$

where $\gamma_\Phi(X^*, Y)$ *is as in Definition 6.5.*

Proof. The hypothesis implies that $\theta = \theta'$, so $u = v$. By Corollary 7.9,

$$J_{\nu^{1/2}}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u, v) \quad \text{and} \quad J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u, v) = J_{\nu^{1/2}\,\mathrm{sgn}_{\epsilon\varpi}}(u, v),$$

so (9.2) agrees with (9.3). We shall work with (9.3), since it is simpler.

By Remark 4.7, $W(G, T_\theta) = \{1, \sigma_\theta\}$, where $\mathrm{Ad}^*(\sigma_\theta)X^* = -X^*$.

We may take the square root w of uv in Proposition 7.5 to be just u. By (8.10),

$$(*) \qquad q^{-m/4} = q^{-(r+1)/2} q^{\operatorname{ord}(s)/2} = q^{-(r+1)/2}\left|s\right|^{-1/2}.$$

By Notations 5.2 and 8.6,

$$(**) \qquad \Phi'_{\varpi^{r+1}}(\pm 2w) = \Phi'(\pm 2s\theta) = \Phi(\pm 2\beta s\theta) = \Phi(\pm\langle X^*, Y\rangle)$$

(the last equality following, for example, from (8.5)).

Suppose that $\operatorname{ord}(s) \not\equiv r \pmod 2$, so that $\gamma_{\mathrm{un}}(s) = 1$ and $\gamma_\Phi(X^*, Y) = 1$. By Corollary 7.9, since $u = v$, (9.3) (hence also (9.2)) becomes

$$\tag{$\dagger$} \begin{aligned} M^G_{X^*}(Y) &= \tfrac{1}{2}\bigl|s\bigr|^{-1/2} q^{-(r+1)/2} \cdot 2 \cdot J^\epsilon_{\nu^{1/2}}(u, v) \\ &= \bigl|s\bigr|^{-1/2} q^{-(r+1)/2} J_{\nu^{1/2}}(u, v). \end{aligned}$$

Since $m > 2$ and $4 \mid m$ by (8.10), combining Proposition 7.5, $(*)$, and $(**)$ gives

$$\tag{$\dagger\dagger$} \begin{aligned} J_{\nu^{1/2}}(u, v) &= q^{-(r+1)/2}\bigl|s\bigr|^{-1/2}\bigl(\Phi(\langle X^*, Y\rangle) + \Phi(-\langle X^*, Y\rangle)\bigr) \\ &= q^{-(r+1)/2}\bigl|s\bigr|^{-1/2} \sum_{\sigma \in W(G, T_\theta)} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, Y\rangle) \\ &= q^{-(r+1)/2}\bigl|s\theta'\bigr|^{-1/2}\gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G, T_\theta)} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

The result (in this case) now follows from Lemma 5.9 by combining ($\dagger$) and ($\dagger\dagger$).

Suppose now that $\operatorname{ord}(s) \equiv r \pmod 2$, so that $\gamma_{\mathrm{un}}(s) = -1$ and

$$\gamma_\Phi(X^*, Y) = \begin{cases} 1, & \theta = 1 \\ -1, & \theta = \epsilon. \end{cases}$$

Again by Corollary 7.9, since $u = v$, (9.3) (hence also (9.2)) becomes (as in ($\dagger$))

$$\tag{$\dagger'$} M^G_{X^*}(Y) = \bigl|s\bigr|^{-1/2} q^{-(r+1)/2}\gamma_{\mathrm{ram}}(s) J_{\nu^{1/2}\operatorname{sgn}_\varpi}(u, v).$$

Since $4 \nmid m$ by (8.10), if $m > 2$, then combining Proposition 7.5, $(*)$, (9.4), and $(**)$ gives (as in ($\dagger\dagger$))

$$\tag{$\dagger\dagger'_{<r}$} \begin{aligned} &J_{\nu^{1/2}\operatorname{sgn}_\varpi}(u, v) \\ &\quad = q^{-(r+1)/2}\bigl|s\theta'\bigr|^{-1/2}\gamma_{\mathrm{ram}}(s)^{-1}\gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G, T_\theta)} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

If $m = 2$, then, by Lemma 5.9, (8.9), and (8.10), $\bigl|s\bigr| = q^{-r}$ and $\operatorname{ord}(u) = -1$. Thus, combining Corollary 7.8, (9.4), and $(**)$ gives

$$\tag{$\dagger\dagger'_{=r}$} \begin{aligned} &J_{\nu^{1/2}\operatorname{sgn}_\varpi}(u, v) \\ &\quad = q^{-1/2}\gamma_{\mathrm{ram}}(s)^{-1}\gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G, T_\theta)} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, Y\rangle) \\ &\quad = q^{-(r+1)/2}\bigl|s\theta'\bigr|^{-1/2}\gamma_{\mathrm{ram}}(s)^{-1}\gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G, T_\theta)} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

The result (in this case) now follows by combining ($\dagger'$), ($\dagger\dagger'_{<r}$) *or* ($\dagger\dagger'_{=r}$), and Lemma 5.9 ∎

9.2. **Close to zero.**

Theorem 9.7. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) > 0$, *and* X^* *is split or unramified, then let* $\gamma_\Phi(X^*, Y)$ *and* $c_0(X^*)$ *be as in Definitions 6.5 and 6.10, respectively. Then*

$$M^G_{X^*}(Y) = c_0(X^*) + \frac{2}{n(X^*)} q^{-(r+1)}\bigl|D_{\mathfrak{g}}(Y)\bigr|^{-1/2}\gamma_\Phi(X^*, Y),$$

where

$$n(X^*) = \begin{cases} 1, & X^* \text{ split} \\ 2, & X^* \text{ elliptic.} \end{cases}$$

Proof. By (8.10), $m < 2$.

By Proposition 8.13, using Notation 8.12, (9.2) becomes

$$M^G_{X^*}(Y) = \tfrac{1}{2}\big[Q_3(q^{-1/2}) + Q_3(q^{-1/2}) - q^{-1} - q^{-1}; 1 + \operatorname{sgn}_\epsilon + \operatorname{sgn}_\varpi + \operatorname{sgn}_{\epsilon\varpi}\big](\theta').$$

Since

$$Q_3(q^{-1/2}) + Q_3(-q^{-1/2}) = -2T^2\big|_{T=q^{-1/2}} = -2q^{-1}, \tag{9.8}$$

this simplifies (by the Plancherel formula on $k^\times/(k^\times)^2$!) to

$$M^G_{X^*}(Y) = [-2q^{-1}; 2, 0, 0, 0].$$

Similarly, (9.3) becomes

$$M^G_{X^*}(Y) = \tfrac{1}{2}\Big(\tfrac{1}{2}(1 + \gamma_{\mathrm{un}}(s))\,\underbrace{\big[Q_3(q^{-1/2}) + \gamma_{\mathrm{un}}(s)Q_3(-q^{-1/2}); 1 + \gamma_{\mathrm{un}}(s)\operatorname{sgn}_\epsilon\big]}_{\text{(I)}} +$$

$$\tfrac{1}{2}(1 - \gamma_{\mathrm{un}}(s))\,\underbrace{\big[-(1 - \gamma_{\mathrm{un}}(s))q^{-1}; (1 - \gamma_{\mathrm{un}}(s)\operatorname{sgn}_\epsilon)\operatorname{sgn}_\varpi\big]}_{\text{(II)}}\Big)(\theta').$$

Since $\gamma_{\mathrm{un}}(s) = \pm 1$ (see Definition 6.5), we may replace $\gamma_{\mathrm{un}}(s)$ by 1 in (I) and by -1 in (II), then use (9.8) and check case-by-case to see that the formula simplifies to

$$M^G_{X^*}(Y) = [-q^{-1}; 1, \gamma_{\mathrm{un}}(s), 0, 0](\theta').$$ □

10. Ramified orbital integrals

Throughout this section, we have

$$\theta = \varpi, \quad \text{so that} \quad r' = r + \tfrac{1}{2} =: h. \tag{10.1}$$

Then $J^\varpi_\chi = J^\varpi_{\chi\operatorname{sgn}_\varpi}$ for $\chi \in \widehat{k^\times}$, so Proposition 8.11 gives

(10.2)
$$M^G_{X^*}(Y) = \tfrac{1}{2}\big|s\big|^{-1/2}\big((1 + \gamma_{\mathrm{ram}}(s))J^\varpi_{\nu^{1/2}}(u, v) + \gamma_{\mathrm{un}}(s)(1 - \gamma_{\mathrm{ram}}(s))J^\varpi_{\nu^{1/2}\operatorname{sgn}_\epsilon}(u, v)\big).$$

By (6.8),

$$\operatorname{sgn}_\varpi(v)G_\varpi(\Phi'_{\varpi^{h+1}}) = \operatorname{sgn}_\varpi(-\varpi)\gamma_{\mathrm{ram}}(s) = \gamma_{\mathrm{ram}}(s). \tag{10.3}$$

10.1. **Far from zero.** As in §9.1, the results of this section are special cases of [55, Proposition VIII.1].

Theorem 10.4. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) < 0$ *and* X^* *is ramified, then* $M^G_{X^*}(Y) = 0$ *unless* X^* *and* Y *lie in* G*-conjugate tori.*

Proof. By (8.10), $m > 2$, so Proposition 7.5 and (8.8) show that $M^G_{X^*}(Y) = 0$ unless $\varpi\theta' \in (k^\times)^2$. By §4, it therefore suffices to consider the case when $-1 \in (\mathfrak{f}^\times)^2$ (so $\operatorname{sgn}_\varpi(-1) = 1$) and $\theta' = \epsilon^2\varpi$, i.e., X^* and Y lie in stably, but not rationally, conjugate tori.

By (8.7), we may take the square root w of uv to be $w = \varpi^{-h}s\epsilon = \epsilon^{-1}u$. Then $u^{-1}w = \epsilon^{-1}$, so Proposition 7.5 shows (whether or not 4 divides m) that, if χ is mildly ramified *and* trivial at -1, then

$$J_{\chi\,\mathrm{sgn}_\varpi}(u,v) = \mathrm{sgn}_\varpi(u^{-1}\varpi)J_\chi(u,v) = -J_\chi(u,v),$$

hence that $J^\varpi_\chi(u,v) = 0$. In particular, this equality holds for $\chi = \nu^{1/2}$ and $\chi = \nu^{1/2}\,\mathrm{sgn}_\epsilon$. It follows from (10.2) that $M^G_{X^*}(Y) = 0$. □

Theorem 10.5. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) < 0$*, and* X^* *and* Y *lie in a common ramified torus* $\mathbf{T}$ *(with* $T = \mathbf{T}(k)$*), then*

$$M^G_{X^*}(Y) = q^{-(h+1)}\left|D_{\mathfrak{g}}(Y)\right|^{-1/2}\gamma_\Phi(X^*,Y)\sum_{\sigma\in W(G,T)}\Phi(\langle \mathrm{Ad}^*(\sigma)(X^*),Y\rangle),$$

where $\gamma_\Phi(X^*,Y)$ *is as in Definition 6.5.*

Proof. Since we have fixed $\theta = \varpi$, the hypothesis implies that $\theta' = \varpi$. In particular, $u = v$. Write σ_ϖ for the non-trivial element of $W(\mathbf{G},\mathbf{T}_\varpi)(k_\varpi)$, so that $\mathrm{Ad}^*(\sigma_\varpi)X^* = -X^*$. Note that it is possible that σ_ϖ is not k-rational. More precisely, by §4, we have that

$$W(G,T_\varpi) = \begin{cases}\{1,\sigma_\varpi\}, & \mathrm{sgn}_\varpi(-1) = 1\\ \{1\}, & \mathrm{sgn}_\varpi(-1) = -1.\end{cases}$$

By (8.10),

$$q^{-m/4} = q^{-(h-\mathrm{ord}(s))/2} = q^{-h/2}\left|s\right|^{-1/2}. \tag{$*$}$$

By Corollary 7.9, since $u = v$,

$$J^\varpi_{\nu^{1/2}}(u,v) = J^\varpi_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u,v),$$

so (10.2) becomes

$$M^G_{X^*}(Y) = \tfrac12\left|s\right|^{-1/2}q^{-(h+1)/2}\big((1+\gamma_{\mathrm{ram}}(s)) + \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big)J^\varpi_{\nu^{1/2}}(u,v). \tag{$\dagger$}$$

It remains to compute $J^\varpi_{\nu^{1/2}}(u,v)$.

We will use Proposition 7.5, but, for simplicity, we want to avoid splitting into cases depending on whether or not $4 \mid m$. By (8.10), the restrictions to $k \setminus \wp^{h-1}$ of $\frac12(1+(-1)^h\,\mathrm{sgn}_\epsilon) = \frac12(1-\gamma_{\mathrm{un}})$ and $\frac12(1+\gamma_{\mathrm{un}})$ are characteristic functions that indicate whether $4 \mid m$ or $4 \nmid m$, respectively. (We omit $\wp^{h-1}$ because we are concerned with the case where $\mathrm{d}(Y) < r$, so that $\mathrm{ord}(s) < r - \frac12 = h-1$.)

Thus, if $\operatorname{sgn}_\varpi(-1) = -1$, then combining Proposition 7.5, $(*)$, and (10.3) gives

$$(***_{\mathrm{ns}})\quad \begin{aligned} J_{\nu^\alpha}(u, v) &= q^{-h/2}\bigl|s\bigr|^{-1/2} \times \\ &\qquad \Bigl(\tfrac{1}{2}\bigl[(1 - \gamma_{\mathrm{un}}(s)) + (1 + \gamma_{\mathrm{un}}(s))\gamma_{\mathrm{ram}}(s)\bigr] \times \\ &\qquad\qquad \Phi'_{\varpi^{h+1}}(2\varpi^{-h}s) \overset{(\S)}{+} \\ &\qquad \tfrac{1}{2}\bigl[(1 - \gamma_{\mathrm{un}}(s)) \overset{(\P)}{-} (1 + \gamma_{\mathrm{un}}(s))\gamma_{\mathrm{ram}}(s)\bigr] \times \\ &\qquad\qquad \Phi'_{\varpi^{h+1}}(-2\varpi^{-h}s)\Bigr) \\ &= \tfrac{1}{2}q^{-(h+1)/2}\bigl|s\theta'\bigr|^{-1/2} \times \\ &\qquad \Bigl(\bigl[(1 + \gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1 - \gamma_{\mathrm{ram}}(s))\bigr]\Phi(\langle X^*, Y\rangle) + \\ &\qquad\qquad \bigl[(1 - \gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1 + \gamma_{\mathrm{ram}}(s))\bigr]\Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle)\Bigr) \end{aligned}$$

and (changing the sign at (§), but not at (¶)) that

$$\begin{aligned} J_{\nu^\alpha \operatorname{sgn}_\varpi}(u, v) &= \tfrac{1}{2}q^{-(h+1)/2}\bigl|s\theta'\bigr|^{-1/2} \times \\ &\qquad \Bigl(\bigl[(1 + \gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1 - \gamma_{\mathrm{ram}}(s))\bigr]\Phi(\langle X^*, Y\rangle) - \\ &\qquad\qquad \bigl[(1 - \gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1 + \gamma_{\mathrm{ram}}(s))\bigr]\Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle)\Bigr), \end{aligned}$$

so that

$$(\ddagger_{\mathrm{ns}})\quad J_{\nu^\alpha}^{\varpi}(u, v) = \tfrac{1}{2}q^{-(h+1)/2}\bigl|s\theta'\bigr|^{-1/2}\bigl[(1 + \gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1 - \gamma_{\mathrm{ram}}(s))\bigr]\Phi(\langle X^*, Y\rangle).$$

Similarly, if $\operatorname{sgn}_\varpi(-1) = 1$, then (changing the sign at (¶), but not at (§), in $(***_{\mathrm{ns}})$) we obtain

$$(***_{\mathrm{s}})\quad \begin{aligned} J_{\nu^\alpha}(u, v) &= J_{\nu^\alpha \operatorname{sgn}_\varpi}(u, v) \\ &= \tfrac{1}{2}q^{-(h+1)/2}\bigl|s\theta'\bigr|^{-1/2}\bigl[(1 + \gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1 - \gamma_{\mathrm{ram}}(s))\bigr] \times \\ &\qquad \bigl[\Phi(\langle X^*, Y\rangle) + \Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle)\bigr], \end{aligned}$$

so that

$$(\ddagger_{\mathrm{s}})\quad \begin{aligned} J_{\nu^\alpha}^{\varpi}(u, v) = J_{\nu^\alpha}(u, v) &= \tfrac{1}{2}q^{-(h+1)/2}\bigl|s\theta'\bigr|^{-1/2}\bigl[(1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\bigr] \times \\ &\qquad \bigl[\Phi(\langle X^*, Y\rangle) + \Phi(\langle \mathrm{Ad}^*(\sigma_\varpi)X^*, Y\rangle)\bigr]. \end{aligned}$$

We may write $(\ddagger_{\mathrm{ns}})$ and $(\ddagger_{\mathrm{s}})$ uniformly as

$$(\ddagger)\quad \begin{aligned} J_{\nu^\alpha}^{\varpi}(u, v) &= \tfrac{1}{2}q^{-(h+1)/2}\bigl|s\theta'\bigr|^{-1/2}\bigl[(1 + \gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1 - \gamma_{\mathrm{ram}}(s))\bigr] \times \\ &\qquad \sum_{\sigma \in N_G(T_\varpi)/T_\varpi} \Phi(\langle \mathrm{Ad}^*(\sigma)X^*, Y\rangle). \end{aligned}$$

Upon combining (†), (‡), and Lemma 5.9, we obtain the desired formula by noting that

$$\begin{aligned}\big[(1+\gamma_{\mathrm{ram}}(s)) + \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big] \cdot \big[(1+\gamma_{\mathrm{ram}}(s)) - \gamma_{\mathrm{un}}(s)(1-\gamma_{\mathrm{ram}}(s))\big] \\ = (1+\gamma_{\mathrm{ram}}(s))^2 - \gamma_{\mathrm{un}}(s)^2(1-\gamma_{\mathrm{ram}}(s))^2 = 4\gamma_{\mathrm{ram}}(s) = 4\gamma_\Phi(X^*, Y)\end{aligned}$$

(since $\gamma_{\mathrm{un}}(s)^2 = 1$). □

10.2. **The bad shell.** We shall be concerned in this section with the behaviour of $M^G_{X^*}$ (hence $\hat\mu^G_{X^*}$, by Proposition 11.2) at the 'bad shell', i.e., on those regular, semisimple elements Y such that $\mathrm{d}(Y) = r$. We shall assume that this is the case throughout the section. By (8.10), this implies that $m = 2$ and that $\mathrm{ord}(\theta')$ is odd, i.e., that Y belongs to a ramified torus. By §4, we can in fact assume that $\mathrm{ord}(\theta') = 1$. Then, by Lemma 5.9,

$$\mathrm{ord}(s) = h - 1 \implies \mathrm{sgn}_\epsilon(s) = (-1)^{h-1} \text{ and } \left|s\theta'\right| = q^{-h}. \tag{10.6}$$

By Definition 6.5, the formula that holds in the situation of Theorem 10.9 holds also, suitably understood, in the situation of Theorem 10.8. We find it useful to separate them anyway.

Remark 10.7. In this section only, we need to name the specific ramified torus in which we are interested. We therefore assume in Theorems 10.8 and 10.9 that $X^* \in \mathfrak{t}^*_\varpi$. See Remark 6.9 for a discussion of how to handle other ramified tori.

Theorem 10.8. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) = 0$*, and* Y *lies in a ramified torus that is* not *stably conjugate to* $\mathbf{T}_\varpi$*, then*

$$M^G_{X^*}(Y) = \tfrac{1}{2}q^{-(h+1)} \cdot q^{-1/2}\left|D_{\mathfrak{g}}(Y)\right|^{-1/2} \sum_{Z \in (\mathfrak{t}_\varpi)_{r:r+}} \Phi(\langle X^*, Z\rangle)\,\mathrm{sgn}_\varpi(Y^2 - Z^2),$$

where we identify the scalar matrices Y^2 *and* Z^2 *with elements of* k *in the natural way.*

Proof. By §4, it suffices to consider the case where $\theta' = \epsilon\varpi$.

By Corollary 7.9, since $\mathrm{ord}(u) = \mathrm{ord}(v)$,

$$J^\varpi_{\nu^{1/2}}(u, v) = J^\varpi_{\nu^{1/2}\,\mathrm{sgn}_\epsilon}(u, v);$$

and, by Corollary 7.8 and (8.8), $J_{\nu^{1/2}\,\mathrm{sgn}_\varpi}(u, v) = 0$, so

$$J^\varpi_{\nu^{1/2}}(u, v) = \tfrac{1}{2}J_{\nu^{1/2}}(u, v);$$

so, by (10.2) and (10.6),

$$\begin{aligned} M^G_{X^*}(Y) &= \tfrac{1}{4}\left|s\right|^{-1/2}q^{-(h+1)/2} \times \\ &\qquad \big((1+\gamma_{\mathrm{ram}}(s)) - (-1)^h\,\mathrm{sgn}_\epsilon(s)(1-\gamma_{\mathrm{ram}}(s))\big)J_{\nu^{1/2}}(u, v) \\ &= \tfrac{1}{4}\left|s\right|^{-1/2}q^{-(h+1)/2} \cdot 2 \cdot J_{\nu^{1/2}}(u, v) \\ &= \tfrac{1}{2}\left|s\right|^{-1/2}J_{\nu^{1/2}}(u, v). \end{aligned} \tag{$*$}$$

Finally, another application of Corollary 7.8, together with (8.9), gives that

$$J_{\nu^{1/2}}(u, v) = q^{-1} \sum_{c \in \wp^{-1}/R} \Phi'_{\varpi^{h+1}}(2c)\,\mathrm{sgn}_\varpi(c^2 - (\varpi^{-h}s)^2\epsilon)$$

Replacing c by $\varpi^{-h}c$, and using (10.6) again, allows us to re-write

$$(**)\qquad J_{\nu^{1/2}}(u,v) = q^{-(h+2)/2}\left|s\theta'\right|^{-1/2} \sum_{c\in\wp^{h-1}/\wp^h} \Phi(2\beta\varpi c)\operatorname{sgn}_\varpi(c^2 - s^2\epsilon).$$

By Definition 4.9, the isomorphism $c \mapsto c\cdot\sqrt{\varpi}$ of k with $\mathfrak{t}_\varpi$ identifies $\wp^{h-1}/\wp^h$ with $(\mathfrak{t}_\varpi)_{(h-1/2):(h+1/2)} = (\mathfrak{t}_\varpi)_{r:r+}$. If c is mapped to Z, then (by (8.5), for example) $2\beta\varpi c = \langle X^*, Z\rangle$, and

$$\operatorname{sgn}_\varpi(c^2 - s^2\epsilon) = \operatorname{sgn}_\varpi(s^2\epsilon\varpi - c^2\varpi) = \operatorname{sgn}_\varpi(Y^2 - Z^2).$$

Combining this observation with $(*)$, $(**)$, and Lemma 5.9 yields the desired formula. $\square$

Theorem 10.9. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) = 0$, *and* $\widetilde{Y}$ *is a stable conjugate of* Y *that lies in a torus with* X^*, *then*

$$\begin{aligned} M^G_{X^*}(Y) = \tfrac{1}{2}q^{-(h+1)}\left|D_{\mathfrak{g}}(Y)\right|^{-1/2} \times &\\ \Big(\gamma_\Phi(X^*,Y) \sum_{\sigma\in W(\mathbf{G},\mathbf{T}_\varpi)} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, \widetilde{Y}\rangle) +&\\ q^{-1/2}\sum_{\substack{Z\in(\mathfrak{t}_\varpi)_{r:r+}\\ Z\neq\pm\widetilde{Y}}} \Phi(\langle X^*, Z\rangle)\operatorname{sgn}_\varpi(Y^2 - Z^2)\Big),& \end{aligned}$$

where $\gamma_\Phi(X^*,Y)$ *is as in Definition 6.5.*

Proof. Implicit in the statement is the hypothesis that $\mathfrak{t} = \mathfrak{t}_{\theta'}$ is stably conjugate to $\mathfrak{t}_\varpi$, so that, by §4, we have $\theta' = x^2\varpi$ for some $x \in R^\times$. The proof proceeds much as in Proposition 10.8.

By (10.6) and Corollary 7.9, since $\operatorname{ord}(u) = \operatorname{ord}(v)$, (10.2) becomes

$$(*)\qquad M^G_{X^*}(Y) = \left|s\right|^{-1/2} q^{-(h+1)/2} J^\varpi_{\nu^{1/2}}(u,v).$$

By (8.7), we may take the square root w of uv to be $w = \varpi^{-h}xs$.

Combining Corollary 7.8 with (8.7), (8.9), and (10.6) gives

$$\begin{aligned} J_{\nu^\alpha}(u,v) &= q^{-1}\sum_{\substack{c\in\wp^{-1}/R\\ c\neq\pm\varpi^{-h}xs}} \Phi'_{\varpi^{h+1}}(2c)\operatorname{sgn}_\varpi(c^2 - (\varpi^{-h}xs)^2)\\ &= q^{-1}\sum_{\substack{c\in\wp^{h-1}/\wp^h\\ c\neq\pm xs}} \Phi(2\beta\varpi c)\operatorname{sgn}_\varpi(c^2 - x^2s^2)\\ &= q^{-(h+2)/2}\left|s\theta'\right|^{-1/2}\sum_{\substack{c\in\wp^{h-1}/\wp^h\\ c\neq\pm xs}} \Phi(2\beta\varpi c)\operatorname{sgn}_\varpi(c^2 - x^2s^2). \end{aligned}$$

Note that $Y^2 = s^2\theta' = x^2s^2\varpi$, and that

$$\widetilde{Y} := xs\sqrt{\varpi} = \operatorname{Ad}\begin{pmatrix}\sqrt{x} & 0\\ 0 & \sqrt{x}^{-1}\end{pmatrix} Y$$

is a stable conjugate of Y that lies in $\mathfrak{t}_\varpi$. (Here, $\sqrt{\varpi}$ is an element of $\mathfrak{g}$, but $\sqrt{x}$ is an element of an extension field of k.) As in Theorem 10.8, if $Z = c\cdot\sqrt{\varpi}$, then

$\langle X^*, Z\rangle = 2\beta\varpi c$ and $\operatorname{sgn}_\varpi(c^2 - x^2 s^2) = \operatorname{sgn}_\varpi(Y^2 - Z^2)$. That is, upon using again the bijection $\wp^{h-1}/\wp^h \to (\mathfrak{t}_\varpi)_{r:r+}$ given by $c \mapsto c \cdot \sqrt{\varpi}$, we obtain

$$(**_1) \qquad J_{\nu^{1/2}}(u, v) = q^{-(h+2)/2}\bigl|s\theta'\bigr|^{-1/2} \sum_{\substack{Z \in (\mathfrak{t}_\varpi)_{r:r+} \\ Z \neq 0, \pm\widetilde{Y}}} \Phi(\langle X^*, Z\rangle) \operatorname{sgn}_\varpi(Y^2 - Z^2).$$

Similarly, combining Corollary 7.8 with (8.9), Lemma 5.9, and (10.3) gives

$$(**_\varpi) \qquad \begin{aligned} J_{\nu^{1/2}\operatorname{sgn}_\varpi}(u, v) &= q^{-1/2}\gamma_{\mathrm{ram}}(s)\bigl(\Phi(2\beta\varpi x s) + \Phi(-2\beta\varpi x s)\bigr) \\ &= q^{-(h+1)/2}\bigl|s\theta'\bigr|^{-1/2}\gamma_{\mathrm{ram}}(s) \sum_{\sigma \in W(\mathbf{G},\mathbf{T})} \Phi(\langle \operatorname{Ad}^*(\sigma)X^*, \widetilde{Y}\rangle). \end{aligned}$$

Combining $(*)$, $(**_1)$, $(**_\varpi)$, and Lemma 5.9 gives the desired formula. □

10.3. **Close to zero.**

Theorem 10.10. *If $\mathrm{d}(X^*) + \mathrm{d}(Y) > 0$, and X^* is ramified, then let $\gamma_\Phi(X^*, Y)$ and $c_0(X^*)$ be as in Definitions 6.5 and 6.10, respectively. Then*

$$M^G_{X^*}(Y) = c_0(X^*) + q^{-(h+1)}\bigl|D_{\mathfrak{g}}(Y)\bigr|^{-1/2}\gamma_\Phi(X^*, Y).$$

Proof. By (8.10), $m < 2$.

By Proposition 8.13 and (6.7), using Notation 8.12, (10.2) becomes

$$\begin{aligned} M^G_{X^*}(Y) = \tfrac{1}{2}\Bigl(&\tfrac{1}{2}(1 + \gamma_{\mathrm{ram}}(s))\bigl[Q_3(q^{-1/2}) + \gamma_{\mathrm{ram}}(s)q^{-1}; 1 + \gamma_{\mathrm{ram}}(s)^{-1}\operatorname{sgn}_\varpi\bigr] + \\ &\tfrac{1}{2}(1 - \gamma_{\mathrm{ram}}(s))\bigl[-Q_3(-q^{-1/2}) + \gamma_{\mathrm{ram}}(s)q^{-1}; (1 - \gamma_{\mathrm{ram}}(s)^{-1}\operatorname{sgn}_\varpi)\operatorname{sgn}_\epsilon\bigr]\Bigr)(\theta'). \end{aligned}$$

By (9.8) and the fact that

$$Q_3(q^{-1/2}) - Q_3(-q^{-1/2}) = -2T(T^2 + 1)\bigr|_{T = q^{-1/2}} = -2q^{-3/2}(q + 1),$$

we may check case-by-case to see that this simplifies to

$$M^G_{X^*}(Y) = \bigl[-\tfrac{1}{2}q^{-3/2}(q + 1); 1, 0, \gamma_{\mathrm{ram}}(s), 0\bigr](\theta').$$ □

11. An integral formula

Remember that all our efforts so far have focussed on the computation of the function $M^G_{X^*}$ of Definition 8.4, whereas we are really interested in the function $\hat{\mu}^G_{X^*}$ of Notation 5.7. We are now in a position to show that they are actually equal.

Lemma 11.1. *If $f \in L^1(G)$, then*

$$\int_G f(g)dg = \int_{k_\theta^\times}\int_k f\bigl(\varphi_\theta^{-1}(\alpha, t)\bigr)\mathrm{d}t\,\mathrm{d}^\times\alpha.$$

In the preceding lemma, $\mathrm{d}g$, $\mathrm{d}t$, and $\mathrm{d}^\times\alpha$ are Haar measures on the obvious groups. Given any two of them, the third can be chosen so that the identity is satisfied. Since Definition 5.4 requires a measure on $G/C_G(X^*)$, not on G, we do not spend much time here worrying about normalisations (although a specific one is used in the proof).

Proof. With respect to the co-ordinate charts $(a,b,c) \mapsto \begin{pmatrix} a & b \\ c & (1+bc)/a \end{pmatrix}$ (for $a \neq 0$) on G and $(a,b,t) \mapsto (a + b\sqrt{\theta}, t)$ on $k_\theta^\times \times k$, the Jacobian of φ_θ at $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (with $a \neq 0$) is $a^{-1} N_\theta(\alpha)$, where $\varphi_\theta(g) = (\alpha, t)$.

In particular, the Haar measure

$$\bigl|a\bigr|^{-1} \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c$$

on G is carried to the measure

$$\bigl|N_\theta(a + b\sqrt{\theta})\bigr|^{-1} \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}t = \bigl|N_\theta(\alpha)\bigr|^{-1} \mathrm{d}\alpha\, \mathrm{d}t = \mathrm{d}^\times\alpha\, \mathrm{d}t$$

on $k_\theta^\times \times k$, as desired. $\square$

Proposition 11.2. *If $X^* \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$ are regular and semisimple, then*

$$\hat{\mu}^G_{X^*}(Y) = M^G_{X^*}(Y),$$

where $M^G_{X^*}$ is as in Definition 8.4, and the Haar measure $\mathrm{d}\dot{g}$ on $G/C_G(X^*)$ of Notation 5.3 is normalised so that

$$\operatorname{meas}_{\mathrm{d}\dot{g}}(\dot{K}) = \begin{cases} q^{-1}(q+1), & X^* \text{ split} \\ q^{-1}(q-1), & X^* \text{ unramified} \\ \frac{1}{2} q^{-2}(q^2-1), & X^* \text{ ramified,} \end{cases}$$

where $\dot{K}$ is the image in $G/C_G(X^*)$ of $\mathrm{SL}_2(R)$.

Proof. We will maintain Notation 5.1. In particular, $X^* \in \mathfrak{t}_\theta^*$.

By the explicit formulæ of the previous sections (specifically, Theorems 9.5, 9.6, 9.7, 10.4, 10.5, 10.8, 10.9, and 10.10), $M^G_{X^*} \in C^\infty(\mathfrak{g}^{\mathrm{rss}})$. This result plays the role of [3, Corollary A.3.4]; we now imitate the proof of Theorem A.1.2 *loc. cit.*

If $f \in C_c(\mathfrak{g}^{\mathrm{rss}})$, then there is a lattice $\mathcal{L} \subseteq \mathfrak{g}$ such that $f \cdot M^G_{X^*}$ is invariant under translation by $\mathcal{L}$. Then

$$\int_{\mathfrak{g}} f(Y) M^G_{X^*}(Y) \mathrm{d}Y = \operatorname{meas}_{\mathrm{d}Y}(\mathcal{L}) \sum_{Y \in \mathfrak{g}/\mathcal{L}} f(Y) \cdot ⨍_{k_\theta^\times / C_\theta} ⨍_k \Phi(\langle X^*, Y \rangle_{\alpha,t}) \mathrm{d}t\, \mathrm{d}^\times \dot{\alpha}.$$

Since the sum is finitely supported, we may bring it inside the integral. By (8.5) and Definition 5.5,

$$(*) \qquad \begin{aligned} &\int_{\mathfrak{g}} f(Y) M^G_{X^*}(Y) \mathrm{d}Y \\ &\quad = ⨍_{k_\theta^\times / C_\theta} ⨍_k \operatorname{meas}_{\mathrm{d}Y}(\mathcal{L}) \sum_{Y \in \mathfrak{g}/\mathcal{L}} f(Y) \Phi\bigl(\langle \mathrm{Ad}^*(\varphi_\theta^{-1}(\alpha,t)) X^*, Y \rangle\bigr) \mathrm{d}t\, \mathrm{d}^\times \dot{\alpha} \\ &\quad = ⨍_{k_\theta^\times / C_\theta} ⨍_k \int_{\mathfrak{g}} f(Y) \Phi\bigl(\langle \mathrm{Ad}^*(\varphi_\theta^{-1}(\alpha,t)) X^*, Y \rangle\bigr) \mathrm{d}Y\, \mathrm{d}t\, \mathrm{d}^\times \dot{\alpha} \\ &\quad = ⨍_{k_\theta^\times / C_\theta} ⨍_k \hat{f}\bigl(\mathrm{Ad}^*(\varphi_\theta^{-1}(\alpha,t)) X^*\bigr) \mathrm{d}t\, \mathrm{d}^\times \dot{\alpha}, \end{aligned}$$

where φ_θ is as in Definition 8.2.

On the other hand, again by Definition 5.5,

$$\hat{\mu}^G_{X^*}(f) := \mu^G_{X^*}(\hat{f}) = \int_{G/T_\theta} \hat{f}\big(\operatorname{Ad}^*(g)X^*\big)\mathrm{d}\dot{g}$$
$$= \int_{\overline{U}\backslash G/T_\theta} \int_{\overline{U}} \hat{f}\big(\operatorname{Ad}^*(\overline{u}g)X^*\big)\mathrm{d}\overline{u}\,\mathrm{d}\ddot{g},$$

where $\overline{U} = \big\{\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right) : b \in k\big\}$. By Lemmata 11.1 and 8.3, and $(*)$, if $\mathrm{d}\dot{g}$ is properly normalised, then

$$\hat{\mu}^G_{X^*}(f) = \int_{k_\theta^\times/C_\theta} \int_k \hat{f}\big(\operatorname{Ad}^*(\varphi_\theta^{-1}(\alpha, t))X^*\big)\mathrm{d}t\,\mathrm{d}^\times\dot{\alpha} = \int_{\mathfrak{g}} f(Y)M^G_{X^*}(Y)\mathrm{d}Y.$$

It remains only to compute the normalisation of $\mathrm{d}\dot{g}$. We do so case-by-case. If X^* is split, so that we may take $\theta = 1$, then the image under φ_1 of

$$(1 + \wp_1) \times \wp \subseteq k_1^\times \times k$$

is precisely the kernel K_+ of the (component-wise) reduction map $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathfrak{f})$. Here, we have written $1 + \wp_1 = \big\{(a, b) \in k_1 : a \in 1 + \wp, b \in \wp\big\}$. Thus,

$$(1 + \wp_1)C_1/C_1 \times \wp \xrightarrow{\sim} K_+T_1/T_1.$$

Now $N_1 : 1 + \wp_1 \to 1 + \wp$ is surjective, so, by Definitions 2.1 and 8.4, the measure (in $k_1/C_1 \times k$) of the domain is

$$\operatorname{meas}_{\mathrm{d}^\times x}(1 + \wp) \cdot \operatorname{meas}_{\mathrm{d}x}(\wp) = q^{-2}.$$

Since $\dot{K} = \mathrm{SL}_2(R)T_1/T_1$ is tiled by

$$\big[\mathrm{SL}_2(R)T_1 \colon K_+T_1\big] = \big[\mathrm{SL}_2(R) \colon K_+(T_1 \cap \mathrm{SL}_2(R))\big] = \big[\mathrm{SL}_2(\mathfrak{f}) \colon \mathsf{T}_1(\mathfrak{f})\big] = q(q + 1)$$

copies of K_+T_1/T_1, where $\mathsf{T}_1 := \big\{\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) : a^2 - b^2 = 1\big\}$, we see that, in this case, $\mathrm{d}\dot{g}$ assigns $\dot{K}$ measure $q^{-2} \cdot q(q + 1) = q^{-1}(q + 1)$.

The remaining cases are easier, since C_θ is contained in the ring R_θ of integers in k_θ, and (for our choices of θ) T_θ is contained in $\mathrm{SL}_2(R)$. If X^* is unramified, so that we may take $\theta = \epsilon$, then the image under φ_ϵ of $R_\epsilon^\times \times R$ is precisely $\mathrm{SL}_2(R)$. Since $N_\epsilon : R_\epsilon^\times \to R^\times$ is surjective, we see that, in this case, $\mathrm{d}\dot{g}$ assigns $\dot{K}$ measure $\operatorname{meas}_{\mathrm{d}^\times x}(R_\epsilon^\times) \cdot \operatorname{meas}_{\mathrm{d}x}(R) = q^{-1}(q - 1)$.

If X^* is ramified, so that we may take $\theta = \varpi$, then the image under φ_ϖ of $R_\varpi^\times \times \wp$ is precisely the Iwahori subgroup $\mathcal{I}$, i.e., the pre-image in $\mathrm{SL}_2(R)$ of $\mathsf{B}(\mathfrak{f}) := \big\{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) : a \in \mathfrak{f}^\times, b \in \mathfrak{f}\big\}$ under the reduction map $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathfrak{f})$. Since $N_\varpi : R_\varpi^\times \to R^\times$ has co-kernel of order 2, we see that, in this case, $\mathrm{d}\dot{g}$ assigns $\dot{K}$ measure

$$\tfrac{1}{2}\operatorname{meas}_{\mathrm{d}^\times x}(R^\times) \cdot \operatorname{meas}_{\mathrm{d}x}(\wp) \cdot \big[\mathrm{SL}_2(\mathfrak{f}) \colon \mathsf{B}\big] = \tfrac{1}{2}q^{-2}(q^2 - 1). \qquad \square$$

In particular, all the results we have proven for $M^G_{X^*}$ are actually results about $\hat{\mu}^G_{X^*}$. We close by summarising some results that can be stated in a fairly uniform fashion (i.e., mostly independent of the 'type' of X^*, in the sense of Definition 4.4). This theorem does *not* cover everything we have shown about Fourier transforms of semisimple orbital integrals (in particular, it says nothing about the behaviour of ramified orbital integrals on the 'bad shell', as in §10.2); for that, the reader should refer to the detailed results of §§9–10.

Theorem 11.3. *If* $\mathrm{d}(X^*) + \mathrm{d}(Y) < 0$ *(or* X^* *is split or unramified and* $\mathrm{d}(X^*) + \mathrm{d}(Y) \le 0$*), then*

$$\hat{\mu}^G_{X^*}(Y) = q^{-(r'+1)} \left|D_{\mathfrak{g}}(Y)\right|^{-1/2} \gamma_\Phi(X^*, Y) \sum_{\sigma \in W(G,T)} \Phi(\langle \mathrm{Ad}^*(\sigma) X^*, Y\rangle)$$

if X^* *and* Y *lie in a common torus* $\mathbf{T}$ *(with* $T = \mathbf{T}(k)$*), and*

$$\hat{\mu}^G_{X^*}(Y) = 0$$

if X^* *and* Y *do not lie in* G*-conjugate tori. Here,* r' *is as in Notation 5.2, and* $\gamma_\Phi(X^*, Y)$ *is as in Definition 6.5.*

If $\mathrm{d}(X^*) + \mathrm{d}(Y) > 0$*, then*

$$\hat{\mu}^G_{X^*}(Y) = c_0(X^*) + q^{-(r'+1)} \left|D_{\mathfrak{g}}(Y)\right|^{-1/2} \gamma_\Phi(X^*, Y).$$

Here, $\gamma_\Phi(X^*, Y)$ *and* $c_0(X^*)$ *are as in Definitions 6.5 and 6.10, respectively.*

Proof. This is an amalgamation of parts of Theorems 9.5, 9.6, 9.7, 10.4, 10.5, and 10.10, and Proposition 11.2. □

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